A STUDY OF GENERATING FUNCTIONS

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By

Shailja Sengar

Under the Supervision of

Dr. R.C. Singh Chandel

Reader & Head, Department of Mathematics

D.V. POSTGRADUATE COLLEGE, ORAI-285001, U.P.

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Dr. R.C. Singh Chandel
M.Sc., Ph.D., M.N.A.Sc., India

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Reader & Head of Mathematics Department
D.V.Postgraduate College
Orai-285001, U.P., India

STD : 05162
Resi : 251607
Off. : 252214
e.mail:<rc_chandel@yahoo.com
Past President :
Rotary Club of Orai
Residence :
4, PROFESSORS COLONY
Rath Road, Orai-285001
U.P., INDIA

DATE 1/12/2006

CERTIFICATE

This is to certify that ***Mrs. Shailja Sengar*** actually carried out the work described in this thesis under my supervision at ***D.V. Postgraduate College, Orai (Jalaun) U.P.*** . She has put the required attendance in the Department of Mathematics during the period of her investigations.

RC Chandel
(Dr. R.C. Singh Chandel)
Reader & Head Department of Mathematics
D.V. Postgraduate College, Orai, Jalaun
Uttar Pradesh

PREFACE

The present work is the outcome of the researches carried out by me in the field of ***"A Study of Generating Functions"*** at D.V. Postgraduate College, Orai, U.P., India. under the supervision of Dr. R.C. Singh Chandel.

The thesis consists nine Chapters. Each Chapter is divided into several sections (Progressively numbered as 1.1, 1.2,....). The formulae are numbered progressively within each Section, namely (5.7.2) refers to equation (2) of Section 7 of Chapter V. References are given in an alphabetical order at the end of every Chapter.

ACKNOWLEDGEMENT

I wish to record my deepest and sincerest feelings of gratitude to ***Dr. R.C. Singh Chandel***, Reader and Head Department of Mathematics. **D.V. Postgraduate College, Orai-285001, U.P.** for his able guidance and expert supervision during the present research work.

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Date

Shailja Sengar
(Shailja Sengar)

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INTRODUCTION

The Chapter I, on Intrdouction presents a breif historical development of the work done in the field of ***"Generating Functions of Special Functions"***. No attempt has been made to give a comprehensive review of the entire literature on the subject but only those aspects, which have a direct bearing on our work done in the present thesis, have been dealt with in some details. It also presents the scope for under taking researches and their importance in related areas.

1.1. Special Functions. An equation of the form

(1.1.1) $p_0(x)\omega^n + p_1(x)\omega^{n-1} + \ldots + p_n(x) = 0$,

where $p_0(x)$, $p_1(x)$,....,$p_n(x)$ are polynomials expressions having integral coefficients, is called algebraic equation. The roots of the above equation

(1.1.2) $w = f(x)$

are called algebraic functions. The functions, which are not roots of algebraic equations are called *"Transcendental Functions"*. Logrithmic functions, exponential functions, trigonometrical functions etc. are examples of "Transcendental Functions". Transendental functions are generally solutions of differntial eqautions or they have integral representations. Transcendental functions such as beta functions, gamma functions, Bessel functions, *E*, *G* and *H*-functions, all polynomials etc., which are of complicated nature are known as *"Higher Transcendental Functions."*

In the study of Higher transcendental functions, if we are not concerned with their general properties, but only with the properties of the functions which occur in the solution of special problems, they are called *"Special Functions"*. Moreover, it is a matter of opinion or

convension. According to Harry-Bateman (1882-1946) any function which has received individual atention at least in one research paper, may be attributed to *Special Function*.

Here we shall discuss some special functions, particularly, polynomials and their generalizations. We shall also discuss the multiple hypergeometric functions of several variables and their applications.

1.2 Legendre Function. Special Functions were first introduced towards the end of eighteenth century in the solutions of the problems of Dynamical Astronomy and Mathematical Physics. In 1782, Laplace introduced the potential theorem. Legendre (1782 or earlier) investigated the expansion of potential function in the form of an infinite series and was thus led to the discovery of functions now known as "*Legendre Coefficients*" or Legendre polynomials.

Thomson and Tait in their well known "*Natural Philosophy*" (1879) defined spherical harmonics as follows :

Any function V_n of Laplace equation $\nabla^2\phi = 0$, which is homogeneous of degree n in x,y,z is called a "Solid Spherical Harmonics of Degree n". The degree n may be any positive integer and the function need not be rational.

If x,y,z are expressed in terms of polar coordinates (r,θ,ϕ) the solid spherical harmonics of degree n assumes the form $r^n f_n(\theta,\phi)$. The function $f_n(\theta,\phi)$ is called a ***"Surface Spherical Harmonics of Degree n"***. Laplace equation possesses solutions of the form $\begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} e^{im\phi} \mathit{H}(\mu)$ where $\mathit{H}(\mu)$ satisfies the ordinary differential equation

$$(1.2.1)\quad (1-\mu^2)\frac{d^2 \mathit{H}}{d\mu^2} - 2\mu\frac{d\mathit{H}}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} \mathit{H} = 0.$$

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restricted to be a real and to be in the interval (-1,1).

Legendre polynomials were generalized by Gegenbauer Tchebicheff and Jacobi. Jacobi polynomials are most general polynomials of this family and were first introduced by C.G. Jacobi in 1859.

Jacobi polynomials (See Reinville [172, p254, (1)]) are defined as

$$(1.2.2)\quad P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,1+\alpha+\beta+n; & \frac{1-x}{2}\\ 1+\alpha; & \end{matrix}\right].$$

For $\alpha=\beta=0$ the above polynomials reduce to Legendre polynomials.

Generating function for Legendre polynomials is given by Rainville ([172,p.157(1)]) :

$$(1.2.3)\quad (1-2xt+t^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n,$$

while their Rodrigues' formula is given by Rainville ([172, p.162(7)])

$$(1.2.4)\quad P_n(x)=\frac{1}{2^n n!}D^n(x^2-1).$$

1.3 Hermite Polynomials. Hermite polynomials, first of all were discussed by Laplace in his two works: *"Treatise on Celestial Mechanics"* ([136], 1805) and *"Theory of Probability"* ([137], 1820). The systematic study of these polynomials was made by C.H. Hermite [109]. Hermite polynomials occur in case of the motion of the point mass in a field of force.

Generating function for Hermite polynomials is given by Rainvile ([172],p.187, (1)])

$$(1.3.1)\quad exp(2xt-t^2)=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}$$

and their Rodrigues' formula is given by Rainville ([172,p.189, (2)])

$$(1.3.2)\quad H_n(x)=(-1)^n exp(x^2)D^n exp(-x^2).$$

1.4 Laguerre Polynomials. E.de Laguerre [135] introduced Laguerre polynomials in 1879. These polynomials occur in case of the motion of two particles (nucleus and electron) that are attached to each other by a force that depends only on the distance between them.

These polynomials satisfy the following differential equation :

$$(1.4.1)\quad x\frac{d^2}{dx^2}L_n^{(\alpha)}(x)+(1+\alpha-x)\frac{d}{dx}L_n^{(\alpha)}(x)+nL_n^{(\alpha)}(x)=0.$$

The generating function for Laguerre polynomials is given by Rainville ([172 p.209(1)])

$$(1.4.2)\quad \sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n=(1-t)^{-1-\alpha}e^{-xt/(1-t)},$$

while their Rodrigues' formula is given by Rainville [172, p.205(5)])

$$(1.4.3)\quad L_n^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{n!}D^n\left[e^{-x}x^{\alpha+n}\right] .$$

1.5 Other Polynomials. There are several hypergeometric polynomials which are non-orthogonal. In 1936 Bateman [9] was interested in constructing inverse Laplace tranforms. For this purpose he introduced the polynomials

$$(1.5.1)\ Z_n(x)={}_2F_2(-n,n+1;1,1;x).$$

Rice [173] made a considerable study of the polynomials defined by

$$(1.5.2)\quad H_n(\zeta,p,v)={}_3F_2\left[\begin{matrix}-n,n+1,\xi; \\ p,1;\end{matrix}\ v\right] .$$

Bateman [8] studied the polynomials

$$(1.5.3)\quad F_n(z)={}_3F_2\left[-n,n+1,\frac{1}{2}(1+z);1,1;z\right]$$

quite extensively, and which were generalized by Pasternak in the following way :

(1.5.4) $F_n^{(m)}(z) = F\left[\begin{matrix} -n, n+1, \frac{1}{2}(1+z+n);, 1 \\ 1, \qquad m+1; \end{matrix}\right].$

Another polynomial, in which the interest is concentrated on a parameter, is Mittage-Leffer polynomial

(1.5.5) $g_n(z) = 2z\ {}_2F_1[1-n, 1-z; 2; 2]$

Bateman (1940) generalized the above polynomials in the form :

(1.5.6) $g_n(z,r) = \frac{(-r)_n}{n!}\ {}_2F_1(-n, z; -r; 2).$

Sister Celine (Fesenmyer [101]) concentrated on the polynomials generated by

(1.5.7) $(1-t)^{-1}\ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix} -4xt/(1-t)^2\right]$

$$= \sum_{n=0}^{\infty} {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1,\ldots,a_p; \\ 1, \quad 1/2, b_1,\ldots,b_q; \end{matrix} x\right] t^n.$$

Her polynomials include Legendre polynomials, some special Jacobi, Rice's $H_n(\zeta, p, v)$, Bateman's $Z_n(x), F_n(z)$ and Pasternak's polynomials etc. as special cases.

1.6 Hypergeometric Function of One Variable. The Gaussian Hypergeometric Series. In the study of second order linear differential equations with three regular singular points there arises the function

(1.6.1) ${}_2F_1(a,b;c;z) = {}_2F_1\left[\begin{matrix} a,b; \\ c; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, c \neq 0, -1, -2, \ldots$

The above infinite series obviously reduces to the elementary geometric series

(1.6.2) $\sum_{n=0}^{\infty} z^n = 1+z+z^2+...+z^n+...$

(1.6.3) (i) $a=c$ and $b=1$ (ii) $a=1$ and $b=c$.

Hence it is called hypergeometric series or more precisely, Gauss's hypergeometric series after the famous German mathematician Carl Friedrich Gauss (1777-1855), who in the year 1812 introduced this series into analysis and gave the F-notation for it.

By D' Alembert's ratio test, it is easily seen that the hypergeometric series in (1.6.1) converges absolutely within the unit circle, that is, when $|z|<1$, provided that the denominator parameter c is neither zero nor negative integer. However, we notice if either or both of the numerator parameters a and b in (1.6.1) is zero or negative integer, the hypergeometric series terminates and the series is automatically convergent. Further tests readily show that the hypergeometric series in (1.6.1) when $|z|=1$ (that is, on the unit circle), is

i) absolutely convergent if $Re(c-a-b)>0$,

ii) conditionally convergent if $-1 < Re(c-a-b) \le 0, \quad z \ne 1$.

iii) divergent if $Re(c-a-b) \le -1$.

In case (i), for a number of summation theorems for the hypergeometric series (1.6.1) when z takes on other special values, see Bailey ([7],1935, pp. 9-11). Erdélyi et al. ([96], 1953, pp. 104-105), Slater ([187], 1966, p.243), Luke ([139], 1975, pp. 271-273) and Srivastava-Manocha ([203], 1984, pp.29-31).

Generalized Hypergeometric Series. A natural generalization of above Gaussian Hypergeometric series ${}_2F_1\,(a,b;c;z)$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. The resulting series

(1.6.4) $${}_pF_q\left[\begin{matrix} a_1,...,a_p; \\ b_1,...,b_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} = {}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$$

is known as the generalized Gauss series, or simply, the generalized hypergeometric series. Here p and q are positive integers or zero (interpreting an empty product as 1), we assume that the variable z, the numerator parameters $a_1,\ldots,a_p$, and denominator parameters $b_1,\ldots,b_q$ take on complex values, provided that

(1.6.5) $b_j \neq 0,-1,-2,\ldots;\ j=1,\ldots,q.$

Supposing that none of the numerator parameters is zero or negative integer (otherwise question of convergence will not arise, and with usual restriction (1.6.5) the ${}_pF_q$ series in (1.6.4)

(i) converges for $|z|<\infty$ if $p\leq q$

(ii) converges for $|z|<1$ if $p=q+1$ and

(iii) diverges for all $z, z\neq 0$, if $p>q+1$

Further more, if we set

$$(1.6.5)\quad \omega=\sum_{j=1}^{q} b_j-\sum_{j=1}^{p} a_j,$$

then the series ${}_pF_q$ with $p=q+1$, is

(i) absolutely convergent for $|z|=1$ if $Re(\omega)>0$,

(ii) conditionally convergent for $|z|=1$, $z\neq 1$ if $-1\leq Re(\omega)\leq 0$. and

(iii) divergent for $|z|=1$, if $Re(\omega)\leq -1$.

1.7 A Further Generalization of ${}_pF_q$. An interesting further generalization of the series ${}_pF_q$ is due to Fox [102] and Wright ([223], [224]), who studied asymptotic expansion of the generalized hypergeometric function defined by

$$(1.7.1)\quad {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\ (b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j+A_jn)}{\prod_{j=1}^{q}(b_j+B_jn)}\frac{z^n}{n!}$$

where the coefficients $A_1,\ldots,A_p$ and $B_1,\ldots,B_q$ are positive real numbers

such that

(1.7.2) $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0.$

By comparing (1.6.4) and (1.7.1), we have

(1.7.3) $${}_p\Psi_q\left[\begin{matrix}(a_1,1),\ldots,(a_p,1); \\ (b_1,1),\ldots,(b_q,1);\end{matrix} z\right] = \frac{\prod_{j=1}^{p}(a_j)}{\prod_{j=1}^{q}(b_j)}\ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p; \\ b_1,\ldots,b_q;\end{matrix} z\right].$$

1.8 Hypergeometric Series in Two Variables.

The great success of the hypergeometic series in one variable has stimulated the development of a corresponding theory in two or more variables. Appell [4] has defined four double hypergeometric series F_1, F_2, F_3, F_4 (known as Appell series), analogus to Gauss's ${}_2F_1(a,b;c;z)$. The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [6], which contains an extensive bibliography of all relevant papers upto 1926 (by for example, L. Pochhammer, J. Horn, E.Picard, E. Goursat). See Erdélyi et al. [96, pp. 222-245] for a review of a subsequent work on the subject; see also Bieley ([7], Chapter 9), Slater ([187], Chapter 8) and Exton ([99] pp. 23-28). Horn puts

$$f(m,n)=\frac{F(m,n)}{F'(m,n)}, g(m,n)=\frac{G(m,n)}{G'(m,n)},$$

where F, F', G, G' are polynomials in m, n of respective degrees p, p', q, q', F' is assumed to have factor $m+1$, and G' a factor $n+1$; F and F' have no common factor exccpt possibily, $m+1$; and G and G' have no common factor except possibily $n+1$. The greatest of the four numbers p, p', q, q' is the order of the hypergeometric series. Horn investigated, in particular, the hypergeometric series order two and found that, apart from certain series which are either expressible in terms of

one variable or are products of two hypergometic series, in one variable, there are essentially thirty four distinct convergent series of order two (Horn [111], correction in Borngässer [14]).

Horn Series. Horn [111] defined the ten hypergeometric series in two varibles and denoted them by $G_1, G_2, G_3, H_1, \ldots, H_7$; he thus completed the set of all fourteen possible second order (complete) hypergeómetric series Appell and Kampé de Fériet ([6], p.143 et seq.), see also Erdélyi et al. ([96]. pp. 224-228).

Cofluent Hypergeometric Series in Two Varibles.

Seven confluent forms of the four Appell series were defined by Humbert [113] and he denoted these confluent hypergeometric series in two variables by $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3, E_1, E_2$.

In addition, there exist thirteen confluent forms of the Horn series which are denoted by Horn [111] and Borngässer [14] $\Gamma_1, \Gamma_2, H_1, \ldots, H_{11}$. Thus there are *twenty* possible confluent hypergeometric series in two variables.

The work of Humbert has been described reasonable fully by Appell and Kampé de Fériet ([6]. pp. 124-135), and the definitions and converegence conditions of all these twenty confluent hypergeometric series in two variables are given also in Erdélyi et al. ([96], pp. 225-228).

For more details see Srivastava and Karlsson [204].

Kampé de Fériet Series and its Generalization. Just as the Gaussian series ${}_2F_1$ was generalized to ${}_pF_q$ by increasing the number of numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [128], who defined a general hypergeometric series in two variables (see Appell and Kampé de Fériet [6, p.150 (29)]). The notation introduced by Kampé de Fériet [loc. cit]

for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy ([15], p.112).

A further generalization of the Kampé de Fériet series is due to Srivastava and Daoust ([196], 1969), who indeed defined the extension of the ${}_p\Psi_q$ series (1.8.3) in two variables.

Later on in 1976, a generalization of Kampé de Fériet series is also seen in the literature due to Srivastava and Panda ([202],p.423,(26)) but it is special case of Srivastava and Daoust ([196], 1969).

1.9 Triple Hypergeometric Series.

Lauricella [134, p. 114] introduced *fourteen* complete hypergeometric series in three variables of the second order. He denoted his triple hypergeometric series by the symbols $F_1, F_2, F_3, \ldots, F_{14}$ of which four series F_1, F_2, F_5 and F_9 correspond respectively to the three variable Lauricella series $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ and $F_D^{(3)}$.

The remaining ten series $F_3, F_4, F_6, F_7, F_8, F_{10}, \ldots, F_{14}$ of Lauricella's set apparently fell into oblivion except that there is an isolated appearance of the triple hypergeometric series F_8 in a paper by Mayr [145, p.265] who came across this series while evaluating certain infinite integrals. Saran [177] initiated a systematic study of these ten triple hypergeometric series of Lauricella's set. Saran's notations are $F_E, F_F, F_G, F_K, F_M, F_N, F_p, F_R, F_S$ and F_T for the series $F_4, F_{14}, F_8, F_3, F_{11}, F_0, F_{12}, F_{10}, F_7, F_{13}$ respectivley (see also Chandel [29]).

Srivastava Triple Hypergeometric Series H_A, H_B *and* H_C:

In the course of further investigation of Lauricella's fourteen hypergeometric series in three additional complete triple hypergeometric series of the second order. These three series H_A, H_B and H_C had been neither included in the Lauricella's set, nor were they previously mentioned in the literature. H_C is new and interesting generaliztion of Appell's series.

F_1; H_B generalizes the Appell series F_2, while H_A provides a generalization of both F_1 and F_2.

A unification of Lauricella's *fourteen* hypergeometric series $F_1,...,F_{14}$ and the additional series H_A, H_B, H_C was introduced by Srivastava [189, p.428], who defined general triple hypergeometric sereis.

While transforming Pochhammer's double-loop contour integrals associated with the series F_8 and F_{14} (i.e. F_G and F_F respectively) belonging to Lauricella's set of hypergeometric series in three variables, the two interesting triple hypergeometric series G_A, G_B of Horn's type were encountered by Pandey ([166], pp.115-116). An investigation of the system of partial differential equation associated with the triple hypergeometric series H_C of Srivastava ([188], [190]) led Srivastava [194, p.105 (3:5)] to new series G_C. Other triple hypergeometric series studied in the literature are introduced by Dhawan [93], Samar [176] and Exton ([100]).

1.10 The Quardruple Hypergeometric Functions. Until the Exton [98] defined and examined a few of their properties, no specific study had been made of any hypergeometric function of four variables apart from the four Lauricella's function $F_A^{(4)}, F_B^{(4)}, F_C^{(4)}$ and $F_D^{(4)}$ certain of their limiting cases. On account of the large number of such functions which arise from a systematic study of all the possibilities he restricted himself to those functions which are complete and of the second order and which involve at least one product of the type $(a, k+m+n+p)$, in series representation; k, m, n pare indices of quadruple summation. Exton ([98], [99]) defined twenty one quadruple hypergeometric series. (see also Chandel and Dwivedi [54]).

Recently Sharma and Parihar [181] introduced *eighty three* hypergeometric functions of four variables. It is worthy to note that out

of these eighty three functions, ninteen functions had already been included in the set of 21 functions introduced by Exton ([98], [99]) in different notations (see, Remark due to Chandel and Kumar [65]). Further very recently Chandel, Agrawal and Kumar [43] have also introduced seven more hypergeometric functions of four vriables:

(1.10.1) $F_{A_1}^{(4)}(a_1,a_1,a_2,a_2,a_2,b_1,b_1 b_2;c_2,c_1,c_1,c_3;x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_2)_{m+p+q}(a_1)_{m+n}(b_1)_{n+p}(b_2)_q}{(c_2)_m(c_1)_{n+p}(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.2) $F_{A_2}^{(4)}(a_1,a_1,a_2,a_2,a_2,b_1,b_1 b_2;c_2,c_1,c_1,c_2;x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_2)_{m+p+q}(a_1)_{m+n}(b_1)_{n+p}(b_2)_q}{(c_2)_{m+q}(c_1)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!};$$

(1.10.3) $F_{A_3}^{(4)}(a_1,a_1,a_2,a_2,a_2,b_1,b_1,b_1;c_2,c_1,c_1,c_2;x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_{m+p}(b_1)_{n+p+q}}{(c_2)_{m+q}(c_1)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!};$$

(1.10.4) $F_{B_1}^{(4)}(a_1,a_1,b_1,a_2,b_1,b_2,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(b_1)_{m+p}(b_2)_{n+p+q}(a_2)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!};$$

(1.10.5) $F_{B_2}^{(4)}(a_1,a_1,b_2,a_1,b_2,b_1,b_1,b_1;c_1,c_2,c_3,c_4;x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+q}(b_1)_{m+p}(b_2)_{n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!};$$

(1.10.6) $F_{C_1}^{(4)}(a_1,a_2,a_1,a_1,b_1,b_1,a_2,b_2;c_1,c_1,c_1,c_1;x,y,z,u)$

Chandel and Sharma ([76],[77]) also introduced following ten more hypergeometric functions of four variables

(1.10.8) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p}(b)_{m+q}(c)_{p+q}(d)_n}{(e)_{m+p}(e')_n(e'')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.9) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+q}(b)_{m+p}(c)_{p+q}(d)_n}{(e_1)_m(e_2)_n(e_3)_p(e_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.10) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{n+p+q-m}(b)_{m+p}(c)_n(d)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.11) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{n+p-m}(b)_{m+p+q}(c)_{n+q}(d)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

(1.10.12) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{n+p-m}(b)_{m+p}(c)_{n+q}(d)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

(1.10.13) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{n+p-m+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.14) $G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{n+p-m}(b_1)_{m+q}(b_2)_n(b_3)_p(b_4)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.15) $G_{B_3}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{n+p-m}(b_1)_{n+q}(b_2)_m(b_3)_p(b_4)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.16) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+p+q}(b)_{m+n}(c)_{n+p}(d)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.10.17) $G_{C_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+q}(b)_{m+p}(c)_{n-p}(d)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

In the present thesis, we shall frequently use these above hypergeometric functions of four variables.

1.11 Multiple Hypergeometric Series of Several Variables.

While several authors, for example, Green [104], Hermite [110] and Dedon [94] have discussed what amount to certain specified hypergeometric functions. It was left to Lauricella [134] to approach this topic systematically. Begining with the Appell functions Lauricella proceeded

to define and study the four important functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ which bear his name.

(1.11.1) $F_A^{(n)}(a,b_1,...,b_n;c_1,...,c_n;x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}, \qquad |x_1|+...+|x_n|<1$$

(1.11.2) $F_B^{(n)}(a_1,...,a_n,b_1,...,b_n;c;x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a_1)_{m_1}...(a_n)_{m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}, \qquad |x_1|<1,...,|x_n|<1.$$

(1.11.3) $F_C^{(n)}(a,b;c_1,...,c_n;x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n}(b)_{m_1+...+m_n}}{(c_1)_{m_1}...(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}, \qquad |x_1|^{1/2}+...+|x_n|^{1/2}<1$$

(1.11.4) $F_D^{(n)}(a,b_1,...,b_n;c;x_1,...,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!},$$

$$|x_1|<1,...,|x_n|<1.$$

A number of confluent forms of the above Lauricella's functions denoted by $\phi_2^{(n)}$ and $\psi_2^{(n)}$ exist in the literature (for instance see Erdélyi [97,p.446 (7.2)]; Humbert [115,p.429], see also Appell and Kampé de Fériet [6,p.134 (34)].

(1.11.5) $\psi_2^{(n)}(a;c_1,...,c_n;x_1,...,x_n)$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

(1.11.6) $\phi_2^{(n)}(b_1,\dots,b_n;c;x_1,\dots,x_n)$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

Exton ([99]) introduced following two multiple hypergeometric sereis ${}^{(k)}_{(1)}E_D^{(n)}$ and ${}^{(k)}_{(2)}E_D^{(n)}$ related to Lauricell's $F_D^{(n)}$

(1.11.7) ${}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\dots,b_n;c,c';x_1,\dots,x_n)$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\dots+m_k}(c')_{m_{k+1}+\dots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

(1.11.8) ${}^{(k)}_{(2)}E_D^{(n)}(a,a',b_1,\dots,b_n;c;x_1,\dots,x_n)$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_k}(a')_{m_{k+1}+\dots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

Prompted by this work, Chandel [32] defined and studied the following multiple hypergeometric function closely related to Lauricella's $F_C^{(n)}$:

(1.11.9) ${}^{(k)}_{(1)}E_C^{(n)}(a,a',b;c_1,\dots,c_n;x_1,\dots,x_n)$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_k}(a')_{m_{k+1}+\dots+m_n}(b)_{m_1+\dots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

Intermediate Lauricella's Functions.

By taking commendable idea of interpolution between Lauricella's function, Chandel and Gupta [59] introduced three multiple hypergeometric functions ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$ and ${}^{(k)}F_{BD}^{(n)}$ related to Lauricella's functions.

(1.11.10) ${}^{(k)}F_{AC}^{(n)}(a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b)_{m_1+\ldots+m_k}(b_{k+1})_{m_{k+1}}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},$$

(1.11.11) ${}^{(k)}F_{AD}^{(n)}(a,b_1,\ldots,b_n;c;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

and

(1.11.12) ${}^{(k)}F_{BD}^{(n)}(a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_k}(a_{k+1})_{m_{k+1}}\ldots(a_n)_{m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}.$$

Chandel and Gupta [59] also introduced following five confluent forms of their above series :

(1.11.13) ${}^{(k)}_{(1)}\phi_{AC}^{(n)}(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b)_{m_1+\ldots+m_k}}{(c_1)_{m_1}\ldots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},k\neq n,$$

(1.11.14) ${}^{(k)}_{(2)}\phi_{AC}^{(n)}(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_{k+1})_{m_{k+1}}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0,$$

(1.11.15) ${}^{(k)}_{(1)}\phi^{(n)}_{AD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_1)_{m_1+\ldots+m_k}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n,$$

(1.11.16) ${}^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n,$$

(1.11.17) ${}^{(k)}_{(2)}\phi^{(n)}_{BD}(a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_{k+1}}\ldots(a_n)_{m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0.$$

Prompted by this work Karlsson [128(a)] also introduced the fourth possible intermediate Lauricella function

(1.11.18) ${}^{(k)}F^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b)_{m_{k+1}+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}.$$

Recently, Chandel and Vishwakarma ([84],[85],[86]) introduced and studied many confluent forms of the above series, defined by

(1.11.20) ${}^{(k)}_{(1)}\phi^{(n)}_{CD}(a,b\ c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b)_{m_{k+1}+\ldots+m_n}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0$$

(1.11.21) $\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n$$

(1.11.22) $\,{}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b)_{m_{k+1}+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0, k \neq n$$

(1.11.23) $\,{}^{(k)}_{(4)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b)_{m_{k+1}+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n$$

(1.11.24) $\,{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b)_{m_{k+1}+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0$$

(1.11.25) $\,{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0$$

(1.11.26) ${}^{(k)}_{(2)}\phi^{(n)}_{AD}\left(a, b_1, \ldots, b_n; c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq 0\,.$$

(1.11.27) ${}^{(k)}_{(3)}\phi^{(n)}_{BD}\left(a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_k; c; x_1, \ldots, x_n\right)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}(a_{k+1})_{m_{k+1}}\ldots(a_n)_{m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n\,.$$

(1.11.28) ${}^{(k)}_{(1)}\phi^{(n)}_{BD}\left(a, b_1, \ldots, b_k; c; x_1, \ldots, x_n\right)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n$$

(1.11.29) ${}^{(k)}_{(2)}\phi^{(n)}_{D}\left(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n\right)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n$$

(1.11.30) ${}^{(k)}_{(1)}\phi^{(n)}_{C}\left(a, b; c_1, \ldots, c_n; x_1, \ldots, x_n\right)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}(b_1)_{m_1+\ldots+m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},\ k \neq n\,.$$

The above multiple hypergeometric functions, will be frequently used in our present thesis.

Generalization of Lauricella's Sereis. An interesting unification and generalization of Lauricalla's multiple series $F_A^{(n)}$ and $F_B^{(n)}$ and Horn's

double series H_2 was considered by Erdélyi (1939). He denoted his series by $H_{n,p}$.

Srivastava and Daoust [197, p. 454] (also see Srivastava and Manocha [203, p.64, (18),(19),(20)]) considered a multivariable extension of the series ${}_p\Psi_q$.

1.12 Extension of Most Generalized Hypergeometric Function of Srivastava and Daoust. As natural further generalization of the (Srivastava-Daoust) generalized Lauricella function of several complex variables [197], H-function of two variables of Mittal-Gupta [148] and G-function of two variables of Agarwal [2] (also see Chandel-Agrawal [39]), is given by Srivastava and Panda ([200], p.271, (4.1); [195], p. 121, (1.10)) by means of the multiple contour integral

$$(1.12.1)\; H^{0,\lambda:[\mu',\nu'];\ldots;[\mu^{(r)},\nu^{(r)}]}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\ [(b'):\theta'];\ldots;[(b^{(n)}):\theta^{(n)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)z_1^{\zeta_1}\ldots z_r^{\zeta_r}\,d\zeta_1\ldots d\zeta_r,\ \omega=\sqrt{-1}$$

where

$$(1.12.2)\quad \phi_i(\zeta_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\zeta_i\right)\prod_{j=1}^{\nu^{(i)}}\Gamma\left(1-b_j^{(i)}+\phi_j^{(i)}\zeta_i\right)}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\zeta_i\right)\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma\left(b_j^{(i)}-\phi_j^{(i)}\zeta_i\right)}\qquad \forall i\in\{1,\ldots,r\};$$

$$(1.12.3)\quad \psi(\zeta_1,\ldots,\zeta_r)=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i\right)}{\prod_{j=\lambda+1}^{A}\Gamma\left(a_j-\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i\right)\prod_{j=1}^{C}\Gamma\left(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\zeta_i\right)}$$

an empty product is inerpreted as 1, the coeffiecients $\theta_j^{(i)}, j=1,\ldots,A;\phi_j^{(i)}$,

$j=1,...B^{(i)}$; $\psi_j^{(i)}, j=1,...,C$; $\delta_j^{(i)}, j=1,...,D^{(i)}$ $\forall i \in \{1,....,r\}$ are positive numbers, and $\lambda, \mu^{(i)}, \nu^{(i)}, A, B^{(i)}, C, D^{(i)}$ are integers such that $0 \le \lambda \le A, 0 \le \mu^{(i)} \le D^{(i)}, C \ge 0$, and $0 \le \nu^{(i)} \le B^{(i)}, \forall i \in \{1,....,r\}$. The contour L_i in the complex ξ_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)})$, $j=1,...,\mu^{(i)}$, are to the right, and those of $\Gamma(1-b_j^{(i)} + \phi_j^{(i)}\zeta_i)$, $j=1,...,\nu^{(i)}$ and $\left(1-a_j + \sum_{i=1}^{r} \theta_j^{(i)}\zeta_i\right)$, $j=1,...,\lambda$, to the left, of L_i, the various parameters being so restricted that these poles are all simple and none of them coincide; and with the points $z_i=0$, $\forall i \in \{1,....,r\}$, being tacitly excluded, the multiple integrals in (1.12.1) converges absolutely if

$$(1.12.4) \quad |arg\, z_i| < \frac{1}{2}\pi\Delta_i, \forall i \in \{1,....,r\},$$

where

$$(1.12.5) \quad \Delta_i = -\sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,$$

$$\forall i \in \{1,....,r\}.$$

The above function is most generalized function of several complex variables and it will be used in the Chapters VIII and IX of our thesis.

1.13 Generalization and unified presentation of polynomials. The orthogonal and non-orthogonal polynomials may be generalized in four ways; (i) by suitable generating function (ii) by rodrigues' formula (iii) by recurrence relation or (iv) by differential equation. In the present thesis we shall make appeal to technique (i) only.

(i) By Defining Suitable Generating Function.

The name *"Generating Function"* was first introduced by Laplace [137] in 1812. If a function $F(x,t)$ has a power series (not necessarily convergent) expansion in t, and it is of the form

$$(1.13.1) \quad F(x,t)=\sum_{n=0}^{\infty} a_n f_n(x)\, t^n,$$

where $a_n; n=0,1,2,\ldots$ be specified sequence independent of x and t then $F(x,t)$ is called generating function of $f_n(x)$.

In the study of polynomial sets, there is a great importance of generating functions. For the use of generating functions we may refer to Sheffer [179], Brenke [13], Rainville [171], [172], Huff [116], Truesdell [211], Palas [165], Boas and Buck [12], Zeitlin [226] and Gould-Hopper [106] etc., Recently Mittal ([146], [147]) and Panda [167] have also discovered many interesting and useful generating functions and operational generating functions for a large number of special functions (polynomials) of Laguerre, Hermite, Bessel, Jacobi etc.

Singhal and Srivastva [182] studied a class of bilateral generating functions for certain classical polynomials. Also Srivastava-Lavoie [199] and Srivastava [192] presented a systematic introduction to and several applications of general method of obtaining bilinear, bilateral or mixed multilateral generating functions for a fairly wide veriety of special functions in one, two or more variables, Bhargava [10] used their theorems for obtaining some bilinear, bilateral and mixed multilateral generating functions. For more details of Generating functions see Chandel-Yadava ([80],[82],[85]), Chandel-Sahgal [67] and Srivastava and Manocha [203].

In 1947, Fasenmyer [101] studied the polynomials (called Sister Celine's polynomials) generated by

$$(1.13.2) \quad (1-t)^{-1}\ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; & \dfrac{-4xt}{(1-t)^2} \\ b_1,\ldots,b_q; & \end{matrix}\right] = \sum_{n=0}^{\infty} {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1,\ldots,a_p; & x \\ 1,\ \ 1/2,\ b_1,\ldots,b_q; & \end{matrix}\right] t^n.$$

Her polynomials include as special cases the Legendre polynomials $P_n(1-2x)$, Jacobi polynomials, Rice's $H_n(p,q,x)$, Bateman's polynomials $Z_n(x)$ and $F_n(x)$. For generalized Rice polynomials see Chandel and Pal [66]. Chandel ([25] to [28]) studied the generalized Laguerre polynomials $f_n^c(x,r)$ (and the polynomials related to them) defined by

$$(1.13.3)\quad (1-t)^{-c}\,exp\left[-(r/(1-t))^r\,xt\right]=\sum_{n=0}^{\infty} f_n^c(x,r)t^n.$$

Agrawal [4] introduced the polynomials defined by

$$(1.13.4)\quad (1-pt^q)^{-c}\,exp\left[-\frac{r^r xt}{(1-pt^q)^r}\right]=\sum_{n=0}^{\infty} f_n^c(x;p,q,r)t^n$$

and discussed the polynomials related to them.

Further Panda [167] generalized above polynomials through generating function :

$$(1.13.5)\quad (1-t)^{-c}\,G\left(\frac{xt^s}{(1-t)^r}\right)=\sum_{n=0}^{\infty} g_n^c(x,r,s)t^n.$$

where c is an arbitrary parameter, r is any integer positive or negative, and $s=1,2,3,\ldots$, and

$$G(z)=\sum_{n=0}^{\infty}\gamma_n z^n,\ (\gamma_0\neq 0)\ .$$

Further Sinha [184] (Also see-Corrigendum due to Chandel [34]) studied special case of $g_n^c(x,r)$, when $\gamma_n=\frac{1}{n!},\gamma_0=1$.

For special interest Chandel and Bhargava ([47],[48]) studied an interesting special case of (1.13.5) when $\gamma_n=\frac{(b)_n}{n!}$.

$$(1.13.6)\quad (1-t)^{-c}\left[1-xt^s/(1-t)^r\right]^{-b}=\sum_{n=0}^{\infty}\Gamma_n^{(b,c)}(x,r,s)t^n$$

and introduced their associated polynomials. Chandel and Chandel [52] also introduced a new class of polynomials through their generating function

$$(1.13.7)\quad (1-pt^q)^{-c}\, G\left(xt/(1-pt^q)^r\right) = \sum_{n=0}^{\infty} g_n^c(x,p,q,r)\, t^n$$

and discussed their related polynomials.

The generalization of all polynomials of Louville [141] Legendre [140], Tchebychef (see [205]), Gegenbaur [103], Humbert [114], Pincherle (as stated in [114] and Kinney [129] led Gould [107] to define the polynomials, through generating function:

$$(1.13.8)\quad (C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m,x,y,p,C).$$

where m is positive integer and other parameters are unrestricted in general.

Srivastava [192] considered the class of generalized Hermite polynomials defined by generating function

$$(1.13.9)\quad \sum_{n=0}^{\infty} \gamma_n^{(m)}(x) \frac{t^n}{n!} = G(mxt - t^m).$$

For its special case $G(z)=e^z$, see Chandel [31].

Chandel and Yadava [78] unified the study of above two classes (1.13.8) and (1.13.9) by considering the following generating function for certain polynomial systems:

$$(1.13.10)\quad G(C - mxt + yt^q) = \sum_{n=0}^{\infty} g_n(m,x,y,q,C)\, t^n.$$

Inspired by (1.13.6) and (1.13.8), Chandel and Bhargava [49] introduced a class of polynomials through generating function

$$(1.13.11)\quad |C - mxt + yt^m|^p \left[1 - \frac{r^r x t^s}{(C - mxt + yt^m)^r}\right]^{-q}$$

$$= \sum_{n=0}^{\infty} B_n^{(p,q)}(m,x,y,r,s,c)\, t^n,$$

where m,s are positive integers and other parameters are unrestricted in general. They also studied their related polynomials.

Further, to unify the study of four general classes (1.13.5), (1.13.7), (1.13.8) and (1.13.11) Chandel [35] introduced a class of polynomials through the generating function.

$$(1.13.12)\quad \left(C-mxt+yt^{m}\right)^{p} G\left[\frac{r^{r}xt^{s}}{\left(C-mxt+yt^{m}\right)^{r}}\right]$$

$$=\sum_{n=0}^{\infty} R_{n}^{p}(m,x,y,r,s,C)t^{n},$$

and also discussed its special case when $\gamma_n=(-1)^n/n!$.

Chandel and Dwivedi ([56],[57]) also considered polynomial systems through generating functions

$$\left(C-mxt+yt^{m}\right)^{p} G\left[\frac{r^{r}zt^{s}}{\left(C-mxt+yt^{m}\right)^{r}}\right]$$

and

$$\left(C-mxt+yt^{m}\right)^{p} G\left[\frac{r^{r}xt^{s}}{\left(C-mxt+yt^{m}\right)^{r}}\right];$$

and discussed their special cases and related polynomials.

To further generalize (1.13.10), Chandel and Yadava [79] introduced some polynomial system of several variables by means of generating function

$$(1.13.13)\quad G\left(a_0+a_1x_1t+\ldots+a_mx_mt^{m}\right)=\sum_{n=0}^{\infty} A_{n,m}^{a_0,\ldots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} t^{n}$$

and discussed their special cases.

To further generalize (1.13.12) and the polynomials of Chandel and Dwivedi ([56],[57]), Chandel and Yadava [79] introduced a polynomial system of several variables through generating function

$$(1.13.14)\quad (a_0+a_1x_1t+\ldots+a_mx_mt^m)^p\, G\left[\frac{r^r x_i t^s}{(a_0+a_1x_1t+\ldots+a_mx_mt^m)^r}\right]$$

$$=\sum_{n=0}^{\infty} B_{n,m,p,r,s}^{a_0,\ldots,a_m}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} t^n$$

and discussed their special cases.

Recently Chandel, Agrawal and Kumar [41] introduced a multivariable analogue of Gould-Hopper's polynomials [106], defined by generating function

$$(1.13.15)\quad \sum_{m_1,\ldots,m_n=0}^{\infty} H_{m_1,\ldots,m_n}^{(h,m,v,p)}(x_1,\ldots,x_n)\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}$$

$$= exp\left[h\left(t_1^m+\ldots+t_n^m\right)\right]\left[1+v\left(x_1t_1+\ldots+x_nt_n\right)\right]^p$$

and discussed is generalization through generating function

$$(1.13.16)\quad exp\left\{h\left(t_1^m+\ldots+t_n^m\right)\right\}G\left[v\left(x_1t_1+\ldots+x_nt_n\right)\right]$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty} S_{m_1,\ldots,m_n}^{(h,m,v)}(x_1,\ldots,x_n)\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}$$

Recently Chandel and Sahgal [68] introduced a multivariable analogue of Panda's polynomials [167], through generating function

$$(1.13.17)\quad (1-t_1)^{-c_1}\ldots(1-t_m)^{-c_m}\left[1-\frac{x_1t_1^{s_1}}{(1-t_1)^{r_1}}\cdots\frac{x_mt_m^{s_m}}{(1-t_m)^{r_m}}\right]^{-b}$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty}\Gamma_{n_1,\ldots,n_m}^{(b:c_1,\ldots,c_m;r_1,\ldots,r_m;s_1,\ldots,s_m)}(x_1,\ldots,x_m)t_1^{n_1}\ldots t_m^{n_m}\ .$$

where $b;c_1,\ldots,c_m$ are any parameters, $r_1,\ldots,r_m$ are any integers positive or negative while $s_1,\ldots,s_m$ are positive integers.

They also considered its generalization through generating function

and discussed other special cases.

(1.13.18) $$(1-t_1)^{-c_1}\ldots(1-t_m)^{-c_m}\,G\left[\frac{x_1 t_1^{s_1}}{(1-t_1)^{r_1}}+\ldots+\frac{x_m t_m^{s_m}}{(1-t_m)^{r_m}}\right]$$

$$=\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(c_1,\ldots,c_m;r_1,\ldots,r_m;s_1,\ldots,s_m)}(x_1,\ldots,x_m)t_1^{n_1}\ldots t_m^{n_m}$$

Very recently, Chandel and Sahgal [69] introduced a multivariable analogue of Gould-Hopper's polynomials [106] and Gould polynomials [107] through generating relation

(1.13.19) $$\sum_{n_1,\ldots,n_m=0}^{\infty} P_{n_1,\ldots,n_r}^{(m_1,\ldots,m_r;M_1,\ldots,M_r;h_1,\ldots,h_r;p)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\ldots\frac{t_r^{n_r}}{n_r!}$$

$$=\left(1+m_1x_1t_1+h_1t_1^{M_1}+\ldots+m_rx_rt_r+h_rt_r^{M_r}\right)^p,$$

where $M_1,\ldots,M_r$ are positive integers and $m_1,\ldots,m_r$, $h_1,\ldots,h_r$ are any numbers real or complex independent of variables $x_1,\ldots,x_r$. They also gave following generalization of (1.13.19) through generating relation

(1.13.20) $$\sum_{n_1,\ldots,n_m=0}^{\infty} g_{n_1,\ldots,n_r}^{(m_1,\ldots,m_r;M_1,\ldots,M_r;h_1,\ldots,h_r)}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\ldots\frac{t_r^{n_r}}{n_r!}$$

$$=G\left(m_1x_1t_1+h_1t_1^{M_1}+\ldots+m_rx_rt_r+h_rt_r^{M_r}\right).$$

Tiwari [209] gave another multivariable analogue of Gould and Hopper's polynomials defined by generating relation

(1.13.21) $$\sum_{n_1,\ldots,n_r=0}^{\infty} H_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;m_1,\ldots,m_r;v_1,\ldots,v_r;p)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\ldots\frac{t_r^{n_r}}{n_r!}$$

$$=exp\left[h_1t_1^{m_1}+\ldots+h_rt_r^{m_r}\right]\left[1+v_1x_1t_1+\ldots+v_rx_rt_r\right]^p,$$

where all $|t_i|<1$, h_i, v_i, k_i and p are any real or complex numbers independent of all variables $x_1,\ldots,x_r$, while all m_i are non-negative integers; $i=1,\ldots,r$.

She also gave its generalization, defined by generating relation.

$$(1.13.22)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} R_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;m_1,\ldots,m_r;v_1,\ldots,v_r)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$= exp\left[h_1 t_1^{m_1}+\ldots+h_r t_r^{m_r}\right]G\left[v_1 x_1 t_1+\ldots+v_r x_r t_r\right],$$

where

$$(1.13.23)\quad G(z)=\sum_{n=0}^{\infty}\gamma_n z^n \qquad \gamma_n \neq 0$$

Further motivated by above works Chandel and Tiwari [71] introduced another generalized multivariable analogoue of Could and Hoppers polynomials [106], defined by generating relation:

$$(1.13.24)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} S_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;m_1,\ldots,m_r;v_1,\ldots,v_r;p)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$=\left[1+v_1 x_1 t_1+\ldots+v_r x_r t_r\right]^p G\left(h_1 t_1^{m_1}+\ldots+h_r t_r^{m_r}\right),$$

where $|t_i|<1$ and all parameters h_i, v_i, p are unrestcited in general but independent of all x_i, while m_i are non negative integers: $i=1,\ldots,r$ and $g(z)$ is given by (1.13.24).

Motivated by above work, in the present thesis we shall introduce multidimensional polynomials through thier generating functions and study them in Chapters II and IV).

1.14 Lie Group Theory.

The theory of continuous groups or as they are now called, Lie groups, was developed by Lie [138] in connection with the integration of systems of differential equations, towards the end of 19th century. Giving due recognition to its importance Ince [119] demonstarated in his exhaustive treatise that the heterogeneous mass of knowledge regarding integration of differential equations can be coordinated in a striking way by means of theory of transformation groups. Using this theory it was established ([119], p.104) that for a differential equation in two variables,

invariant under a known group, an integrating factor may atleast theoretically, be found and thus its integration be made possible. However, from the standpoint of special functions, the result which has still greater significance is as follow ([119], p.97). Every one-parametric transformation group G in two variables (x,y) contains a unique infinitesimal transformation U and in terms of U, the fnitie equations of the group are

(1.14.1) $x_1 = (exp\ Ut)$ x and $y_1 = (exp\ Ut)$ y,

t being the paremeter.

Further more, variation in a function $f(x,y)$ under the corresponding finite transformation is given by the eqaution

(1.14.2) $f\ (x_1,y_1) = (exp\ \ Ut)\ f\ (x,y).$

This fundamental result was destined to contribute substantially in application of group theoretic methods to the special functions. But it appears that due consideration to this techinique was not given for quite a long time. The much awaited break was provided by Infeld and Hull, in their classic paper "The factorization method" ([120], 1951). They established that certain classes of second order differential equations can be factorized as product of two first order differential operators, the first order operators giving rise to recurrence relations for eigen functions of those second order equations. This historic research work laid the ground work for Lie theoretic approach to the special functions.

In the factorization method a single second order differential equation is replaced by a pair of equations of the form

(1.14.3) $L_n^+ f_n = f_{n+1}, L_n^- f_n = f_{n-1}$,

where L_n^+ and L_n^- are first order differential operators.

It was found that these first order operators L_n^+, L_n^- together with some additional operatiors can be identified with certain Lie algebras. The

special functions, then, emerged in dual capacity as the basis functions in representation space for these Lie algebras as well as the matrix elements in the induced Lie group representations.

Soon after, taking cue from this method Wigner, Weisner, and Vilenkin, applied Lie theory and group representations to the study of special functions.

Winger, and Inonu ([221], 1953) constructed representations of Euclidean group of the plane E_2 as a limit of representations of the group of rotations in three space, O_3, and obtained elements of representations of the former group (Bessel functions) as limits of representations of the latter group (Jacobi-functions) by method of contraction.

Taking note of the inherent connection between the differential recurrence relations and differential equations satisfied by a variety of special functions, Weisner [218, 219, 220] derived an ingenious method of obtaining generating functions. He started with the ordinary differential equation

$$L\left(x, \frac{d}{dx}, n\right)V = 0 \tag{1.14.4}$$

for a given special function $V_n(x)$ and constructed a partial differential equaiton

$$L\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right)f(x,y) = 0\,. \tag{1.14.5}$$

After this, his method consisted in finding a non-trivial continuous group of transformations under which (1.14.5) remained invariant and using this group to obtain solutions of (1.14.5). These solutions yielded the generating function for the solutions $V_n(x)$ of (1.14.4).

In perspective one can claim without any reservation that *Weisner's*

method has definitely proved to be one of the most effective methods for obtaining generating relations.

Vilenkin [212, 1956] demonstrated that the faithful irreducible unitary representations of Euclidean group in the plane E_3, with respect to a suitable basis, have matrix elements proportional to Bessel functions of integral order and obtained some properties of Jacobi polynomials [213, 1957]. He and his coworkers Akin and Levin [214, 1957] then computed matrix elements of the unitary irreducible representations of E_6, the Euclidean group in 3-space. The matrix elements turned out to be spinor valued solutions of wave equation $(\nabla^2 + w^2)\psi(r) = 0$ and are of wide applicability in theoretical physics. Later on he employed [216, 1968] the irreducible representations of three groups, namely, the group $SO_{(n)}$ of rotations of n dimensional Euclidean space, the group $SH_{(n)}$ of hyperbolic rotations of n dimensional space and the group of motions of $(n-1)$ dimensional Euclidean space to derive new special functions $P^{nl}_{kmj}(\cos\theta), B^{n\sigma}_{kmj}(\cosh\theta)$ and $J^{n}_{kmj}(x)$ which turned out to be generalizations of Jacobi polynomials, conical functions and Bessel functions respectively.

The brilliant researches of these mathematicians received impetus at the hands of Miller and Kaufman. Miller [149, 1964] showed that six distinct types of factorizations classified by Infeld and Hull [120] are obtainable from a study of representation theory of four Lie algebras; the Lie algebra of the Euclidean group in 2 and 3-spaces, the Lie algebra of rotation group in 3-space and a certain 4-dimensional solvable Lie algebra. He also derived recurrence relations and generating functions for hypergeometric, Bessel and parabolic cylindrical functions by representation theory. Then he extended [150, 1968] Lie algebra representations, discussed in [149] to irreducible representations of corresponding Lie groups and obtained the hypergeometric functions as the matrix elements.

Later, choosing the six dimensional algebra T_6 of complex Euclidean group in 3-space, Miller in a series of three papers [152, 153, 154] showed that irreducible representations of corresponding group yielded identities for Whittaker functions, Bessel functions, Gegenbauer polynomials and Jacobi polynomials.

In his quest for achieving unification in the theory of special functions Miller concentrated mostly on Lie algebraic considerations. On the other hand Kaufman, [124, 1966] found it more suitable to desist from overall classification. She took the known families of known special functions and their differential recurrence relations as given. From each set of recurrence relations she generated the corresponding Lie algebras and derived various expansions.

By this time, a marked upsurge in the interest in this field could be witnessed. In 1968, three outstanding books [151], [208] and [216] dealing with generating functions concept from group theoretic point of view were published. These books played a significant role in earning a wider acceptability for the new approach.

Following the method of Weisner, Vishwanathan [217, 1968] derived generating functions for ultraspherical polynomials using a three dimensional Lie algebra.

Orihara, [164, 1966] thought it to be more appropriate to treat Hermite polynomials by representation theory of infinite dimensional group of motions G_∞. Kono, [126, 1966] also derived various known formulae for Hermite polynomials. He showed that matrix elements for representations of G_∞ were the limits of matrix elements of representations of finite dimensional Euclidean groups of motions investigated by Vilenkin [212], [214], [215]).

Rayko [174, 1966] derived some properties of Hankel functions from the representations of group of motions of pseudo Euclidean plane.

Tretjakova [210, 1969] derived explicit formulae for the generating function of the matrix elements of unitary irreducible representations for $SO_{(n)}$ $(n \geq 3)$, discussed earlier by Vilenkin [214]. Stolov [200] derived asymptotic expansions for the classes of functions $P^{nl}_{kmj}(\cos\theta)$, $B^{n\sigma}_{kmj}(\cosh\theta)$, $J^{n}_{kmj}(x)$ introduced by Vilenkin [214]. Cukerman [88, 1974] also continued the work in the same field to arrive at the differential equations for functions $P^{nl}_{kmj}(x)$. By studying the action of certain second order elements of the centre of the enveloping algebra, Rozenbljum, [175, 1975] derived systems of second order differential equation for matrix elements of the group $SO_{(n)}$, $SO_{(n,1)}$ and the group of motion of n-dimensional Euclidean space.

While Russian mathematicians were mainly influenced by works of Vilenkin, the researchers else where concentrated on Weisner's techniques. Defining two infinitesimal differential operators on the basis of recurrence relations obeyed by a given special function, Chatterjea ([20], [21], [22], [23]) designated them as raising and lowering operators as suggested by Kaufman, to obtain generating relations for Laguerre, Hermite, Gegenbauer and simple Bessel polynomials. Das ([89],[90],[91],[92]) also obtained some bilateral generating functions of Jacobi, Hermite, Laguerre and Bessel polynomials by the same method. Adopting more or less same technique Saha ([23], 1977) derived certain generating functions for the generalized Bessel polynomials.

Kyriakopoulos, ([132],[133], 1974) used Lie algebra D_2 with six operators instead of 3-dimensional Lie algebra which was used by Weisner and Miller to treat hypergeometric functions. Wong and Kesarwani ([222],1975) used (p+3) dimensional Lie group K_{p+3} and obtained

identities of the generalised hypergeometric functions ${}_pF_q$. Chiney ([87], 1976) established an addition theorem for associated Legendre functions.

Manocha, [142] and Manocha and Jain [143] made small modifications in Weisner's method to derive several new bilinear and bilateral generating functions. Whereas in classical Weisner's method [218] the Lie group operator was made to act on solutions $f(x,y)=V_n(x)y^n$ of the partial differential equation associated with given special function $V_n(x)$, Manocha choosed $f(x,y)$ as common eigen function of the Casimir operator and of an element of the enveloping algebra of Lie algebra $SL(2)$.

Lie theoretic approach to special functions attained a new dimension through the work of Patera and Winternitz ([168], 1973) when they stepped outside the class of hypergeometric functions to cover Lame and Heun polynomials. Pham Ngoc Dinh, Alain ([169], 1974) extended Patera-Winternitz method to Mathieu functions. Subsequently he derived addition formulae concerning Mathieu, parabolic cylindrical functions and Bessel polynomials [170].

In early 1970's Miller came out with two truly note-worthy papers ([155], 1973; [156], 1974) on the study of generalised hypergeometric functions and Meijer's G-function by representation theory. Starting with an abstract Lie algebra and its realization in terms of Lie derivatives, he obtained generalized hypergeometric functions ${}_pF_q$'s as matrix elements in the corresponding group representation. He also derived a number of formal identities satisfied by these functions using Weisner's method. Later confining himself to the particular case of Gaussian hypergeometric functions ${}_2F_1$ and its dynamical symmetry algebra he used identical techniques to arrive at some important identities for the same. Continuing his work on applications of machinery of representation theory he used dynamical symmetry algebras of various many variable generalizations of hypergeometric functions namely Appell's functions, Lauricella functions

etc. to derive a variety of results for these functions ([157], [158], [159]).

In addition, number of researchers, Ikeda ([117, 1967; [118], 1975), Zalgapin ([225, 1972), Dunkl [95, 1975], Takahashi ([207], 1975), Holman, Biedenharn and Louck ([112], 1976), to name a few, have enriched the literature on the subject by their valuable contributions.

An entirely new field in application of Lie theory to the special functions was opened up by Winternitz and Fritz ([211], 1965), with introduction of a group theoretic method for separation of variables in the principal partial differential equations of mathematical physics. Generalizing their methods Miller, initially alone ([160],[161], 1974) and then in collaboration with Boyer and Kalnins ([16],[17],[18]) has produced a series of papers relating symmetry groups of these linear partial differential equations and the co-ordinates systems in which variables separate for these equations. In these papers they have applied representation theoretic characterization of separation of variables for the derivation of various physically significant identities.

Recently Kalnins, Manocha and Miller ([121],[122],[123]) have carried application of Lie-algebraic technique in study of two variable hypergeometric functions still further. They have demonstrated that the thirty-four functions defined by Horn [111] indeed arise by separation of variables in certain important equations in mathematical physics just as 1-variable hypergeometric functions do.

In the present thesis in Chapter VII, we shall extend general class of generating functions through group-theoretic approach and also make their applications.

1.15 Applications of Special Functions. For applications of Special Functions in mathematical physics for mixed boundary value problems one may refer to Sneddon [183]. Chandel [36] discussed a mixed boundary value problem on heat conduction and

determined the temprature at any point on the surface of sphere by solving dual series equations involving the Legendre polynomials, Chandel-Bhargava [51], Chandel-Dwivedi [58] and Chandel-Yadava [81] dicussed a problem on heat conduction employing generalized Kampé de Fériet funciton of Srivastava-Daoust [196], Srivastava's hypergeometric function of three variables [189], and multiple hypergeometric function of Srivastava-Daoust [197] respectively.

Chandel and Bhargava [50] discussed a problem on cooling of a heated cylinder using generalized Kampé de Fériet function of Srivastava and Daoust [196] Chandel and Gupta [64] used multiple hypergeometric function of Srivastava and Daoust [197] in the solution of a problem on heat conduction in a finite bar, while Chandel and Gupta ([59],[60]) made applications of multivariable H-function of Srivastava and Panda defined by (1.12.1) in the problems of heat conduction and in cooling of a heated cylinder, respectively. Chandel, Agrawal and Kumar [40] used multivariable H-funciton of Srivastava and Panda in problem on electrostatic potential in spherical regions. Further Chandel, Agrawal and Kumar [42] evaluated an integral involving Kampé de Fériet function and multivariable H-function of Srivastava and Panda, and then applied it to solve a problem on a circular disk. Chandel, Agarwal and Kumar [44] also used multivariable H-function of Srivastava and Panda in Fourier series.

Further Chandel, Agrawal and Kumar [45] made application of Lauricella's $F_D^{(n)}$ in determining velocity coefficient of chemical reaction.

Chandel and Tiwari [72] employed mutliple hypergeometric funciton of Srivastava and Daoust ([196],[197]) to solve two boundary value problems on (1) *heat conduction in a rod* (ii) *deflection of vibrating string under certain conditions*. Very recently, Chandel and Singh [75] employed multivariable polynomials of Srivastava [195] and multivariable

H-function of srivastava and Panda ([200],[201]) to solve two boundary value problems under certain conditions. Here in the present thesis in Chapters VIII and IX we employ multivariable *H*-function of Srivastava and Paṇda ([200], [201]) and generalized polynomials of Srivastava [191] in two boundary value problems

1.16. A Breif Survey of the Chapters.

In the Chapter II of the present thesis, we introduce Multivariable generalized polynomials defined through multilinear generating functions.

In the chapter III, we derive generating functions for multiple hypergeometric functions of several variables through operational techniques and their special cases are discussed.

In Chapter IV, we shall introduce some associated polynomials defined through their generating functions.

Chapter V deals with generating relations, Taylor's series and Maclaurine's sereies expansions of hypergeometric functions of four variables.

In Chapter VI, we establish generating relations, Taylor's series and Maclaurine's sereis expansions for multiple hypergeometric functions of several variables.

Chapter VII introduces a general class of generating functions through group theoretic approach and discusses its applications.

Chapters VIII and IX discuss the applications of multivariable *H*-function of Srivastava and Panda and generalized polynomials of Srivastava in Mixed Boundary Value Problems.

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MULTIVARIABLE GENERALIZED POLYNOMIALS DEFINED THROUGH THEIR GENERATING FUNCTIONS

CHAPTER -II

MULTIVARIABLE GENERALIZED POLYNOMIALS DEFINED THROUGH THEIR GENERATING FUNCTIONS

2.1.Introduction. Gould [2] considered a class of generalized Humbert polynomials defined by

$$(2.1.1)\quad (C-mxt+yt^{m})^{p}=\sum_{n=0}^{\infty}P_{n}(m,x,y,p,C)t^{n},$$

where m is a positive integer and other parameters are unresticted in general.

Further in order to unify several hither to considered polynomial systems belonging to (or providing extensions of) the family of classical Jocobi, Hermite and Laguerre polynomials, Panda [3] introduced polynomials $\{g_n^c(x,r,s)|n=0,1,2,...\}$ generated by

$$(2.1.2)\quad (1-t)^{-c}G\left(\frac{xt^{s}}{(1-t)^{r}}\right)=\sum_{n=0}^{\infty}g_{n}^{c}(x,r,s)t^{n},$$

where c is an arbitrary parameter, r is any integer, positive or negative and s=1,2,3,.... .

On the other hand Srivastava [4] considered a class of generalized Hermite polynomials defined by the generating function

$$(2.1.3)\quad \sum_{n=0}^{\infty}\gamma_{m}(x)\frac{t^{n}}{n!}=G(mxt-t^{m}),$$

where and in (2.1.2)

(2.1.4) $G(z)=\sum_{n=0}^{\infty}\gamma_n z^n, \quad \gamma_0 \neq 0$

and m is an arbitrary positive integer.

To unify the study of these three general classes of polynomials defined by (2.1.1), (2.1.2) (2.1.3), Chandel [1] considered a generating function

(2.1.5) $$\left(C-mxt+yt^m\right)^p G\left[\frac{r^r x t^s}{\left(C-mxt+yt^m\right)^r}\right]=\sum_{n=0}^{\infty} R_n^p(m,x,y,r,s,C)t^n$$

where m, s are positive integers, other parameters are unresticted in gernal and $G(z)$ is given by (2.1.4)

Motivated by (2.1.5), here we consider the polynomials

$$\left\{R_{n_1,\ldots,n_m}^{(p_1,\ldots,p_n)}([m],[x],[y],[r],[s],[C]) \mid n_i=0,1,2,\ldots;\ i=1,\ldots,n\right\}$$

of several variables defined through generating function

(2.1.6) $$\left(C_1-mx_1t_1+y_1t_1^{m_1}\right)^{p_1}\ldots\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{p_n}$$

$$G\left[\frac{r_1^{r_1}x_1t_1^{s_1}}{\left(C_1-m_1x_1t_1+y_1t_1^{m_1}\right)^{r_1}}+\ldots+\frac{r_n^{r_n}x_nt_n^{s_n}}{\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{r_n}}\right]$$

$$=\sum_{n_1,\ldots,n_n=0}^{\infty} R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)}([m],[x],[y],[r],[s],[C])t_1^{n_1}\ldots t_n^{n_n},$$

where $G(z)$ is given by (2.1.4) and m_i, s_i $(i=1,\ldots,n)$ are positive integers and other parameters are unresticted in genral.

For $n=1$, (2.1.6) reduces to the linear generating function (2.1.5) due to Chandel ([1], p.186, (1.5)).

For brevity, throughout our investigation we shall write $R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)}([m],[x],[y],[r],[s],[C])$ as $R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)}$ only.

2.2. Explicit form. Starting from generating relation (2.1.6), we

have

$$\sum_{n_1,\ldots,n_n=0}^{\infty} R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)} t_1^{n_1}\ldots t_n^{n_n} = \sum_{K=0}^{\infty}\gamma_k\left[\frac{r_1^{r_1}x_1t_1^{s_1}}{\left(C_1-m_1x_1t_1+y_1t_1^{m_1}\right)^{r_1}}+\ldots+\frac{r_n^{r_n}x_nt_n^{s_n}}{\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{r_n}}\right]^K$$

$$\left(C_1-m_1x_1t_1+y_1t_1^{m_1}\right)^{p_1}\ldots\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{p_n}$$

$$=\sum_{k_1,\ldots,k_n=0}^{\infty}\gamma_{k_1+\ldots+k_n}\frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}\left(r_1^{r_1}x_1\right)^{k_1}\left(C_1-m_1x_1t_1+y_1t_1^{m_1}\right)^{p_1-r_1k_1}\ldots$$

$$\left(r_n^{r_n}x_n\right)^{k_n}\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{p_n-r_nk_n}t_1^{s_1k_1}\ldots t_n^{s_nk_n}$$

$$=\sum_{n_1,\ldots,n_n=0}^{\infty}\ \sum_{k_1,\ldots,k_n=0}^{\infty}\gamma_{k_1+\ldots+k_n}\frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}P_{n_1}(m_1,x_1,y_1,p_1-r_1k_1,C_1)\ldots$$

$$P_{n_n}(m_n,x_n,y_n,p_n-r_nk_n,C_n)\left(r_1^{r_1}x_1\right)^{k_1}\ldots\left(r_n^{r_n}x_n\right)^{k_n}t_1^{n_1+s_1k_1}\ldots t_n^{n_n+s_nk_n}$$

$$=\sum_{n_1,\ldots,n_n=0}^{\infty}\ \sum_{k_1=0}^{[n_1/s_1]}\cdots\sum_{k_n=0}^{[n_n/s_n]}\gamma_{k_1+\ldots+k_n}\frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}P_{n_1-k_1s_1}(m_1,x_1,y_1,p_1-r_1k_1,C_1)\ldots$$

$$P_{n_n-k_ns_n}(m_n,x_n,y_n,p_n-r_nk_n,C_n)r_1^{r_1k_1}x_1^{k_1}\ldots r_n^{r_nk_n}x_n^{k_n}t_1^{n_1}\ldots t_n^{n_n}.$$

Thus equating the coefficients of $t_1^{n_1}\ldots t_n^{n_n}$ both the sides, we derive the explicit form :

$$(2.2.1)\quad R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)}=\sum_{k_1=0}^{[n_1/s_1]}\cdots\sum_{k_n=0}^{[n_n/s_n]}\gamma_{k_1+\ldots+k_n}\frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}P_{n_1-k_1s_1}(m_1,x_1,y_1,p_1-r_1k_1,C_1)\ldots$$

$$P_{n_n-k_ns_n}(m_n,x_n,y_n,p_n-r_nk_n,C_n)r_1^{r_1k_1}x_1^{k_1}\ldots r_n^{r_nk_n}x_n^{k_n},$$

where $P_n(m,x,y,p,c)$ are generalized Humbert polynomials defined by Gould [2] through (2.1.1).

2.3. Other Applications of Generating Relation.

Making an appeal to generating relation (2.1.6), we derive

$$(2.3.1)\ g_{n_1,\dots,n_n}^{(p_1+p'_1,\dots,p_n+p'_n)} = \sum_{k_1=0}^{n_1} \dots \sum_{k_n=0}^{n_n} g_{k_1,\dots,k_n}^{(p'_1,\dots,p'_n)} P_{n_1-k_1}(m_1,x_1,y_1,p_1,C_1)$$

$$\dots P_{n_n-k_n}(m_n,x_n,y_n,p_n,C_n)$$

and

$$(2.3.2)\ R_{n_1,\dots,n_n}^{(p_1,\dots,p_n)} = \sum_{k_1=0}^{n_1} \dots \sum_{k_n=0}^{n_n} R_{n_1-k_1,\dots,n_n-k_n}^{(p'_1,\dots,p'_n)} P_{k_1}(m_1,x_1,y_1,p_1-p'_1,C_1)$$

$$\dots P_{k_n}(m_n,x_n,y_n,p_n-p'_n,C_n).$$

2.4 Recurrence Relations. Making an application to generating relation (2.1.6), we obtain the recurrence relation

$$(2.4.1)\ R_{n_1,\dots,n_n}^{(p_1+1,\dots,p_n+1)} = (c_1+\dots+c_n) R_{n_1,\dots,n_n}^{(p_1,\dots,p_n)} + m_1 x_1 R_{n_1-1,n_2,\dots,n_n}^{(p_1,\dots,p_n)} + \dots +$$

$$m_i x_i R_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_m} + m_n x_n R_{n_1,\dots,n_{n-1},n_n-1}^{(p_1,\dots,p_n)}$$

$$+ y_1 R_{n_1-m_1,n_2,\dots,n_n}^{(p_1,\dots,p_n)} + \dots + y_n R_{n_1,\dots,n_n-m_n}^{(p_1,\dots,p_n)},$$

which for brevity can be written as

$$(2.4.2)\ R_{n_1,\dots,n_n}^{(p_1+1,\dots,p_n+1)} = (c_1+\dots+c_n) R_{n_1,\dots,n_n}^{(p_1,\dots,p_n)} + \sum_{i=1}^{n} m_i x_i R_{n_1,\dots,n_i-1,n_{i+1},\dots,n_n}^{(p_1,\dots,p_n)}$$

$$+ \sum_{i=i}^{n} y_i R_{n_1,\dots,n_{i-1},n_i-m_i,n_{i+1},\dots,n_n}^{(p_1,\dots,p_n)}.$$

2.5. Differential Recurrence Relations. Differentiating (2.1.6) partially with respect to x_1, we have

$$(2.5.1)\ \sum_{n_1,\dots,n_n=0}^{\infty} \frac{\partial}{\partial x_1} R_{n_1,\dots,n_n}^{(p_1,\dots,p_n)} t_1^{n_1} \dots t_n^{n_n} + m_1 p_1 \sum_{n_1,\dots,n_n=0}^{\infty} R_{n_1-1,n_2,\dots,n_n}^{(p_1-1,p_2,\dots,p_n)} t_1^{n_1} \dots t_n^{n_n}$$

$$=\left[C_1-m_1x_1t_1+y_1t_1^{m_1}\right]^{p_1-r_1-1}\left(C_2-m_2x_2t_2+y_2t_2^{m_2}\right)^{p_2}\ldots\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{p_n}$$

$$\left[C_1+r_1^{r_1}t_1^{s_1}+m_1x_1r_1^{r_1}(r_1-1)t_1^{s_1+1}+y_1r_1^{r_1}t_1^{s_1+m_1}\right]G'\ .$$

Also differentiating (2.1.6) partially with respect to t_1, we get

$$(2.5.2)\quad \sum_{n_1,\ldots,n_n=0}^{\infty}(n_1+1)R_{n_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}+m_1x_1p_1\sum_{n_1,\ldots,n_n=0}^{\infty}R_{n_1,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}$$

$$=\left(C_1-m_1x_1t_1+y_1t_1^{m_1}\right)^{p_1-r_1-1}\left(C_2-m_2x_2t_2+y_2t_2^{m_2}\right)^{p_2}\ldots\left(C_n-m_nx_nt_n+y_nt_n^{m_n}\right)^{p_n}$$

$$\left[C_1r_1^{r_1}x_1s_1t_1^{s_1-1}-m_1r_1^{r_1}x_1^2(r_1-1)t_1^{s_1}-x_1y_1r_1^{r_1}(m_1r_1-s_1)t_1^{m_1+s_1-1}\right]G'\ .$$

Eleminating G' from (2.5.1) and (2.5.2), we derive

$$(2.5.3)\left[\cdot\sum_{n_1,\ldots,n_n=0}^{\infty}(n_1+1)R_{n_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}+m_1x_1p_1\sum_{n_1,\ldots,n_n=0}^{\infty}R_{n_1,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}\right]$$

$$\left[C_1r_1^{s_1}t_1^{s_1}+m_1x_1r_1^{r_1}(r_1-1)t_1^{s_1+1}+y_1r_1^{s_1}t_1^{s_1+m_1}\right]$$

$$=\left[\sum_{n_1,\ldots,n_n=0}^{\infty}\frac{\partial}{\partial x_1}R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}+m_1p_1\sum_{n_1,\ldots,n_n=0}^{\infty}R_{n_1,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}\right]$$

$$\left[C_1r_1^{r_1}x_1s_1t_1^{s_1-1}-m_1r_1^{r_1}x_1^2(s_1-1)t_1^{s_1}-x_1y_1r_1^{r_1}(m_1r_1-s_1)t_1^{m_1+s_1-1}\right]\ .$$

Now equating the coefficients of $t_1^{n_1}\ldots t_n^{n_n}$ both the sides we finally derive the differential recurrence realtion

$$(2.5.4)\ c_1r_1(n_1-s_1+1)R_{n_1-s_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}+m_1x_1r_1^{r_1}(r_1-1)(n_1-s_1)R_{n_1-s_1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}$$

$$+y_1r_1^{r_1}(n_1-s_1-m_1+1)R_{n_1-s_1-m_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}+m_1x_1p_1c_1r_1^{r_1}R_{n_1-s_1,n_2,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}$$

$$+m_1^2 x_1^2 r_1^{r_1} p_1 (r_1 - 1) R^{(p_1-1,p_2,\dots,p_n)}_{n_1-s_1-1,n_2,\dots,n_n} + m_1 x_1 y_1 p_1 r_1^{r_1} R^{(p_1-1,p_2,\dots,p_n)}_{n_1-s_1-m_1,n_2,\dots,n_n}$$

$$= C_1 r_1^{r_1} x_1 s_1 \frac{\partial}{\partial x_1} R^{(p_1,\dots,p_n)}_{n_1-s_1+1,n_2,\dots,n_n} - m_1 r_1^{r_1} x_1^2 (s_1 - 1) \frac{\partial}{\partial x_1} R^{(p_1,\dots,p_n)}_{n_1-s_1,n_2,\dots,n_n}$$

$$- x_1 y_1 r_1^{r_1} (m_1 r_1 - s_1) \frac{\partial}{\partial x_1} R^{(p_1-1,p_2,\dots,p_n)}_{n_1-s_1,n_2,\dots,n_n} - m_1^2 r_1^{r_1} x_1^2 p_1 (s_1 - 1) R^{(p_1-1,p_2,\dots,p_n)}_{n_1-s_1-1,n_2,\dots,n_n}$$

$$+ m p_1 c_1 r_1^{r_1} x_1 s_1 R^{(p_1-1,p_2,\dots,p_n)}_{n_1-s_1,n_2,\dots,n_n} - x_1 y_1 r_1^{r_1} m_1 p_1 (m_1 r_1 - s_1) R^{(p_1-1,p_2,\dots,p_n)}_{n_1-m_1-s_1,n_2,\dots,n_n},$$

which further suggests n-differential reccurrence relation in the following unified form :

$$(2.5.5)\; c_i r_i (n_i - s_i + 1) R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i+1,n_{i+1},\dots,n_n} + m_i x_i r_i^{r_i} (r_i - 1)(n_i - s_i) R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$+ y_i r_i^{r_i} (n_i - s_i - m_i + 1) R^{(p_1-1,p_2,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-m_i+1,n_{i+1},\dots,n_n} + c_i r_i^{r_i} m_i x_i p_i R^{(p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$+ n_i^2 x_i^2 r_i^{r_i} p_i (r_i - 1) R^{(p_1,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_n} + m_i x_i y_i p_i r_i^{r_i} R^{(p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-m_i,n_{i+1},\dots,n_n}$$

$$= c_i r_i^{r_i} x_i s_i \frac{\partial}{\partial x_i} R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i+1,n_{i+1},\dots,n_n} - m_i r_i^{r_i} x_i^2 (s_i - 1) \frac{\partial}{\partial x_i} R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$- x_i y_i r_i^{r_i} (m_i r_i - s_i) \frac{\partial}{\partial x_i} R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-m_i-s_i+1,n_{i+1},\dots,n_n} + m_i p_i c_i r_i^{r_i} x_i s_i R^{(p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$- m_i^2 r_i^{r_i} x_i^2 (s_i - 1) R^{(p_1,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n} - x_i y_i r_i^{r_i} m_i p_i (m_i r_i - s_i) R^{(p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-m_i-s_i,n_{i+1},\dots,n_n}$$

$$i = 1,\dots,n.$$

Now further differentiating (2.1.6) partially with respect to y_1, we have

$$(2.5.6)\; p_1 \sum_{n_1,\dots,n_n=0}^{\infty} R^{(p_1-1,p_2,\dots,p_n)}_{n_1-m_1,n_2,\dots,n_n} t_1^{n_1} \dots t_n^{n_n} - \sum_{n_1,\dots,n_n=0}^{\infty} (n_1 + 1) R^{(p_1,\dots,p_n)}_{n_1+1,n_2,\dots,n_n} t_1^{n_1} \dots t_n^{n_n}$$

$$= r_1^{r_1+1} x_1 t_1^{s_1+m_1} \left(C_1 - m_1 x_1 t_1 + y_1 t_1^{m_1}\right)^{p_1-r_1-1} \left(C_2 - m_2 x_2 t_2 + y_2 t_2^{m_2}\right)^{p_2}$$

$$\ldots\left(C_n - m_n x_n t_n + y_n t_n^{m_n}\right)^{p_n} G'.$$

Thus eleminating G' from (2.5.2) and (2.5.6) we obtain

$$(2.5.7)\left[\sum_{n_1,\ldots,n_m=0}^{\infty}(n_1+1)R_{n_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}+m_1x_1p_1\sum_{n_1,\ldots,n_n=0}^{\infty}R_{n_1,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}\right]$$

$$\left[r_1^{s_1+1}x_1t_1^{s_1+m_1}\right]$$

$$=\left[C_1r_1^{r_1}x_1s_1t_1^{s_1-1}-m_1r_1^{r_1}x_1^2(s_1-1)t_1^{s_1}-x_1y_1r_1^{r_1}(m_1r_1-s_1)t_1^{m_1+s_1-1}\right]$$

$$\left[p_1\sum_{n_1,\ldots,n_m=0}^{\infty}R_{n_1-m_1,n_2,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}-\sum_{n_1,\ldots,n_n=0}^{\infty}\frac{\partial}{\partial y_1}R_{n_1,\ldots,n_n}^{(p_1,\ldots,p_n)}t_1^{n_1}..t_n^{n_n}\right].$$

Now equating the coefficients $t_1^{n_1}\ldots t_n^{n_n}$ both the sides, we establish differential recurrence relation

$$(2.5.8)\ r_1(n_1-s_1-m_1+1)R_{n_1-s_1-m_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}+r_1x_1m_1p_1R_{n_1-s_1-m_1,n_2,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}$$

$$=c_1p_1s_1R_{n_1-m_1-s_1+1,n_2,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}-m_1p_1r_1^{r_1}(s_1-1)x_1^2R_{n_1-m_1-s_1,n_2,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}$$

$$-y_1p_1(m_1r_1-s_1)R_{n_1-2m_1-s_1+1,n_2,\ldots,n_n}^{(p_1-1,p_2,\ldots,p_n)}-c_1s_1\frac{\partial}{\partial y_1}R_{n_1-s_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}$$

$$+m_1(s_1-1)x_1\frac{\partial}{\partial y_1}R_{n_1-s_1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}+(m_1r_1-s_1)y_1\frac{\partial}{\partial y_1}R_{n_1-m_1-s_1+1,n_2,\ldots,n_n}^{(p_1,\ldots,p_n)}.$$

which suggests n-differential recurrence relations in the following unified form:

$$(2.5.9)\ r_i(n_i-s_i-m_i+1)R_{n_1,\ldots,n_{i-1},n_i-s_i-m_i+1,n_{i+1},\ldots,n_n}^{(p_1,\ldots,p_n)}+r_ix_im_ip_iR_{n_1,\ldots,n_{i-1},n_i-s_i-m_i,n_{i+1},\ldots,n_n}^{(p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)}$$

$$=c_ip_is_iR_{n_1,\ldots,n_{i-1},n_i-s_i-m_i+1,n_{i+1},\ldots,n_n}^{(p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)}-m_ip_i(s_i-1)x_iR_{n_1,\ldots,n_{i-1},n_i-s_i-m_i,n_{i+1},\ldots,n_n}^{(p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)}$$

$$-y_i p_i (m_i r_i - s_i) R^{(p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-2m_i-s_i+1,n_{i+1},\dots,n_n} - c_i s_i \frac{\partial}{\partial y_i} R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-m_i+1,n_{i+1},\dots,n_n}$$

$$+m_i (s_i - 1) x_i \frac{\partial}{\partial y_i} R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n} + (m_i r_i - s_i) y_i \frac{\partial}{\partial y_i} R^{(p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-m_i-s_i+1,n_{i+1},\dots,n_n}$$

$i=1,\dots,n.$

2.6. Special Case I.

For $\gamma_n = \dfrac{(-1)^n}{n!}$,, our polynomials $R^{(p_1,\dots,p_n)}_{n_1,\dots,n_n}([m],[x],[y],[r],[s],[C])$ are reduced to the polynomials $E^{(p_1,\dots,p_n)}_{n_1,\dots,n_n}([m],[x],[y],[r],[s],[C])$ defined as

(2.6.1) $E^{(p_1,\dots,p_n)}_{n_1,\dots,n_n}([m],[x],[y],[r],[s],[C])$

$$= C^{p_1}_{n_1}(m_1,x_1,y_1,r_1,s_1,C_1)..C^{p_n}_{n_n}(m_n,x_n,y_n,r_n,s_n,C_n),$$

where $C^p_n(m,x,y,r,s,C)$ are generalized polynomials due to Chandel ([1], p.187 (2.3.1)) defined through the generating relation

(2.6.2) $$(C - mxt + yt^m)^p \, exp\left[-\frac{xr^r t^s}{(C - mxt + yt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} C^p_n(m,x,y,r,s,C) t^r$$

as special case of (2.1.5).

2.7 Special Case II. For $\gamma_n = \dfrac{(b)_n}{n!}$, (2.1.6) defines polynomials $B^{(b:p_1,\dots,p_n)}_{n_1,\dots,n_n}([m],[x],[y],[r],[s],[C])$ by (multilinear generating function :

$$(2.7.1)\quad \left(C_1 - m_1x_1t_1 + y_1t_1^{m_1}\right)^{p_1} \ldots \left(C_n - m_nx_nt_n + y_nt_n^{m_n}\right)^{p_n}$$

$$\left[1 - \left\{\frac{r_1^{r_1}x_1t_1^{s_1}}{\left(C_1 - m_1x_1t_1 + y_1t_1^{m_1}\right)^{r_1}} + \ldots + \frac{r_n^{r_n}x_nt_n^{s_n}}{\left(C_n - m_nx_nt_n + y_nt_n^{m_n}\right)^{r_n}}\right\}\right]^{-b}$$

$$= \sum_{n_1,\ldots,n_n=0}^{\infty} B_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_n)}([m],[x],[y],[r],[s],[C])\, t_1^{n_1}\ldots t_n^{n_n} .$$

For brevity through out our investigations we shall write $B_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_n)}$ only, for $B_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_n)}([m],[x],[y],[r],[s],[C])$.

2.8. Explicit Form. Starting from (2.7.1), we derive the following explicit form :

$$(2.8.1)\quad B_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_n)} = \sum_{k_1=0}^{[n_1/s_1]} \ldots \sum_{k_n=0}^{[n_n/s_n]} \frac{(b)_{k_1+\ldots+k_n}}{k_1!\ldots k_n!} P_{n_1-k_1s_1}(m_1,x_1,y_1,p_1-r_1k_1,C_1)$$

$$\ldots P_{n_n-k_ns_n}(m_n,x_n,y_n,p_n-r_nk_n,C_n)\, r_1^{r_1k_1}x_1^{k_1}\ldots r_n^{r_nk_n}x_n^{k_n} .$$

2.9 Other results. An application to generating relation (2.7.1) shows that

$$(2.9.1)\quad B_{n_1,\ldots,n_n}^{(b+b';p_1+p'_1,\ldots,p_n+p'_n)} = \sum_{k_1=0}^{n_1} \ldots \sum_{k_n=0}^{n_1} B_{k_1,\ldots,k_n}^{(b;p_1,\ldots,p_n)} B_{n_1-k_1,\ldots,n_n-k_n}^{(b';p'_1,\ldots,p'_n)} ,$$

$$(2.9.2)\quad B_{n_1,\ldots,n_n}^{(b;p_1+p'_1,\ldots,p_n+p'_n)} = \sum_{k_1=0}^{n_1} \ldots \sum_{k_n=0}^{n_n} B_{n_1-k_1,\ldots,n_n-k_n}^{(b;p_1,\ldots,p_n)} P_{k_1}(m_1,x_1,y_1,p'_1,C_1)$$

$$\ldots P_{k_n}(m_n,x_n,y_n,p'_n,C_n)$$

and

$$(2.9.3)\quad B_{n_1,\ldots,n_n}^{(b+b';p_1,\ldots,p_n)}=\sum_{q_1=0}^{n_1}\cdots\sum_{q_n=0}^{n_1}\sum_{k_1=0}^{[n_1/s_1]}\cdots\sum_{k_n=0}^{[n_n/s_n]}\frac{(b')_{k_1+\ldots+k_n}}{k_1!..k_n!}r_1^{r_1k_1}\ldots r_n^{r_nk_n}x_1^{k_1}\ldots x_n^{k_n}$$

$$B_{n_1-q_1,\ldots,n_n-q_n}^{(b;p_1,\ldots,p_n)}P_{q_1-k_1s_1}(m_1,x_1,y_1-r_1p_1,C_1)..P_{q_n-k_ns_n}(m_n,x_n,y_n-r_np_n,C_n).$$

2.10 Differential Recurrence relation. Differentiating (2.7.1) partially with respect to x_1, and equating the coefficients of $t_1^{n_1}\ldots t_n^{n_n}$ both the sides, we derive

$$(2.10.1)\quad \frac{\partial}{\partial x_1}B_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_n)}+m_1p_1B_{n_1-1,n_2,\ldots,n_n}^{(b;p_1-1,p_2,\ldots,p_n)}+bm_1r_1^{r_1}(r_1+1)x_1B_{n_1-s_1-1,n_2,\ldots,n_n}^{(b+1;p_1-r_1-1,p_2,\ldots,p_n)}$$

$$=bc_1r_1^{r_1}B_{n_1-s_1,n_2,\ldots,n_n}^{(b+1;p_1-r_1-1,p_2,\ldots,p_n)}+by_1r_1^{r_1}B_{n_1-s_1-m_1,n_2,\ldots,n_n}^{(b+1;p_1-r_1-1,p_2,\ldots,p_n)},$$

which suggests n-results in the following unified form :

$$(2.10.2)\quad \frac{\partial}{\partial x_i}B_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_n)}+m_ip_iB_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_n}^{(b;p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)}$$

$$+bm_ir_i^{r_i}(r_i+1)B_{n_1,\ldots,n_{i-1},n_i-s_i-1,n_{i+1},\ldots,n_n}^{(b+1;p_1,\ldots,p_{i-1},p_i-r_i-1,p_{i+1},\ldots,p_n)}$$

$$=bc_ir_i^{r_i}B_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_n}^{(b+1;p_1,\ldots,p_{i-1},p_i-r_i-1,p_{i+1},\ldots,p_n)}+by_ir_i^{r_i}B_{n_1,\ldots,n_{i-1},n_i-s_i-m_i,n_{i+1},\ldots,n_n}^{(b+1;p_1,\ldots,p_{i-1},p_i-r_i-1,p_{i+1},\ldots,p_n)},$$

$i=1,\ldots,n.$

Further differenting (2.7.1) partially with respect to x_1 and equating the coefficient of $t_1^{n_1}\ldots t_n^{n_n}$ both the sides, we obtain

$$(2.10.3)\quad (n_1+1)B_{n_1+1,n_2,\ldots,n_n}^{(b;p_1,\ldots,p_n)}+m_1x_1p_1B_{n_1,\ldots,n_n}^{(b;p_1-1,p_2,\ldots,p_n)}$$

$$+br_1^{s_1}x_1y_1(r_1m_1-s_1)B_{n_1-m_1-s_1+1,n_2,\ldots,n_n}^{(b+1;p_1-r_1-1,p_2,\ldots,p_n)}$$

$$=br_1^{s_1}x_1s_1c_1B_{n_1-s_1+1,n_2,\ldots,n_n}^{(b+1;p_1-r_1-1,p_2,\ldots,p_n)}+br_1^{r_1+1}x_1^2m_1B_{n_1-s_1,n_2,\ldots,n_n}^{(b+1;p_1-r_1-1,p_2,\ldots,p_n)},$$

which suggests n-reccurence relations in the following unified form :

$$(2.10.4)\quad (n_i+1)B_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_n}^{(b;p_1,\ldots,p_n)}+m_ix_ip_iB_{n_1,\ldots,n_n}^{(b;p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)}$$

$$=br_i^{r_i}x_i\Big[s_ic_iB^{(b+1;p_1,\dots,p_{i-1},p_i-r_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_n}+r_ix_im_iB^{(b+1;p_1,\dots,p_{i-1},p_i-r_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$-y_i(r_im_i-s_i)B^{(b+1;p_1,\dots,p_{i-1},p_i-r_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-m_i-s_i,n_{i+1},\dots,n_n}\Big],\quad i=1,\dots,n.$$

Applying the techniques of the result (2.5.4) we also derive the differential recurrence relation :

(2.10.4) $$c_1r_1(n_1-s_1+1)B^{(b;p_1,\dots,p_m)}_{n_1-s_1+1,n_2,\dots,n_n}+m_1x_1r_1^{r_1}(r_1-1)(n_1-s_1)B^{(b;p_1,\dots,p_m)}_{n_1-s_1,n_2,\dots,n_n}$$

$$+y_1r_1^{r_1}(n_1-s_1-m_1+1)B^{(b;p_1,\dots,p_n)}_{n_1-s_1-m_1+1,n_2,\dots,n_n}+c_1r_1^{r_1}m_1x_1p_1B^{(b;p_1-1,p_2,\dots,p_n)}_{n_1-s_1,n_2,\dots,n_n}$$

$$+m_1^2x_1^2r_1^2p_1(r_1-1)B^{(b;p_1-1,p_2,\dots,p_n)}_{n_1-s_1-1,n_2,\dots,n_n}+m_1x_1y_1p_1r_1^{r_1}B^{(b;p_1-1,p_2,\dots,p_n)}_{n_1-s_1-m_1,n_2,\dots,n_n}$$

$$=c_1r_1^{r_1}x_1s_1\frac{\partial}{\partial x_1}B^{(b;p_1,\dots,p_m)}_{n_1-s_1+1,n_2,\dots,n_n}-mr_1^{s_1}x_1^2(s_1-1)\frac{\partial}{\partial x_1}B^{(b;p_1,\dots,p_m)}_{n_1-s_1,n_2,\dots,n_n}$$

$$-x_1y_1r_1^{r_1}(m_1r_1-s_1)\frac{\partial}{\partial x_1}B^{(b;p_1,\dots,p_n)}_{n_1-s_1-m_1+1,n_2,\dots,n_n}+mp_1c_1r_1^{r_1}x_1s_1B^{(b;p_1-1,p_2,\dots,p_n)}_{n_1-s_1,n_2,\dots,n_n}$$

$$-m_1^2r_1^{r_1}x_1^2p_1(s_1-1)B^{(b;p_1-1,p_2,\dots,p_n)}_{n_1-s_1-1,n_2,\dots,n_n}-x_1y_1r_1^{r_1}m_1p_1(m_1r_1-s_1)B^{(b;p_1-1,p_2,\dots,p_n)}_{n_1-s_1-m_1,n_2,\dots,n_n}$$

which suggests m-differential recurrence relations in the following unified form:

(2.10.5) $$c_ir_i(n_i-s_i+1)B^{(b;p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i+1,n_{i+1},\dots,n_n}+m_ix_ir_i^{r_i}(r_i-1)(n_i-s_i)B^{(b;p_1,\dots,p_m)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$+y_ir_i^{r_i}(n_i-s_i-m_i+1)B^{(b;p_1,\dots,p_n)}_{n_1,n_{i-1},n_i-s_i-m_i+1,n_{i+1},\dots,n_n}+c_ir_i^{r_i}m_ix_ip_iB^{(b;p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$+m_i^2x_i^2r_i^{r_i}p_i(r_i-1)B^{(b;p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_m)}_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_n}+m_ix_iy_ip_ir_i^{r_i}B^{(b;p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-m_i,n_{i+1},\dots,n_n}$$

$$=c_ir_i^{r_i}x_is_i\frac{\partial}{\partial x_i}B^{(b;p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i+1,n_{i+1},\dots,n_n}-m_ir_i^{s_i}x_i^2(s_i-1)\frac{\partial}{\partial x_i}B^{(b;p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i,n_{i+1},\dots,n_n}$$

$$-x_iy_ir_i^{r_i}(m_ir_i-s_i)\frac{\partial}{\partial x_i}B^{(b;p_1,\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i-m_i+1,n_{i+1},\dots,n_n}+m_ip_ic_ir_i^{r_i}x_is_iB^{(b;p_1,\dots,p_{i-1},p_i-1,p_{i+1},\dots,p_n)}_{n_1,\dots,n_{i-1},n_i-s_i+n_{i+1},\dots,n_n}$$

$$-m_i^2 r_i^{r_i} x_i^2 p_i (s_i - 1) B_{n_1,\ldots,n_{i-1},n_i-s_i-1,n_{i+1},\ldots,n_n}^{(b;p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)} - x_i y_i r_i^{r_i} m_i p_i (m_i r_i - s_i) B_{n_1,\ldots,n_{i-1},n_i-m_i-s_i,n_{i+1},\ldots,n_n}^{(b;p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_n)};$$

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GENERATING FUNCTIONS THROUGH OPERATIONAL TECHNIQUES

CHAPTER -III

GENERATING FUNCTIONS THROUGH OPERATIONAL TECHNIQUES

The aim of the present Chapter is to ellustrate the application of differential operator $T_K = x(k+xD),\ D \equiv \frac{d}{dx}$ to derive an intersting generating relation for generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12],[13]; also see Srivastava and Karlsson ([14,p.37, eqns (2.1) to (2.3)), with its special cases among others for Lauricella's $F_D^{(n)}$ [8], its confluent form $\phi_D^{(n)}$ and for hypergeometric function of three variables $F^3(x,y,z)$ of Srivastava [10].

3.1 Introduction . Garg [6] used the following operational formula due to Mittal [9] :

$$(3.1.1)\ \sum_{n=0}^{\infty}\frac{z^n}{n!}T^n_{\alpha+1+\beta n}f(x)=\frac{(1+v)^{\alpha+1}}{1-\beta v}f[x(1+v)],$$

where $v = xz(1+v)^{\beta+1}$, β being a constant, $f(x)$ admits a formal power series in x, $T_k = x(k+xD),\ D \equiv \frac{d}{dx}$ and T_k^n means that the operator T_k is repeated n times, in obtaining a known generating relation due to Srivastava and Panda ([15],p.130, eqn. (4.3)) for multivariable H-function of Srivastava and Panda ([15],[16],[17], also see Srivastava, Gupta and Goyal [18]) obtained through quite different non-operational technique.

Our present Chapter shows the importance and utility of the differential operator T_k, in obtaining the generating relation for generalized multiple

hypergeometric function of Srivastava and Daoust ([11],[12],[13]; also see Srivastava and Karlsson [14,p.37, eqn. (2.1) to (2.3)). It also presents its interesting speical cases among others specially for Lauricella's $F_D^{(n)}$ [8], its confluent form $\phi_D^{(n)}$ and for triple hypergeometric series $F^{(3)}$ of Srivastava [10].

3.2. Main Generating Relation . In this Section, we derive the following generating relation for generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12],[13]) by operational technique :

$$(3.2.1)\ \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n F^{A+1:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\ [\alpha+(\beta+1)n:\sigma_1,\ldots,\sigma_r]:\\ [(c):\psi',\ldots,\psi^{(r)}],[\alpha+\beta n:\sigma_1,\ldots,\sigma_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix}\, z_1(1+v)^{\sigma_1},\ldots,z_r(1+v)^{\sigma_r}\right),$$

provided that $v=t(1+v)^{\beta+1}$, $|z_i|<1$, $\sigma_i>0$, and

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,\quad i=1,\ldots,r.$$

Proof. In (3.1.1), choosing

$$(3.2.2)\ f(x)=F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\ [(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(c):\psi',\ldots,\psi^{(r)}],[(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix}\, y_1x^{\sigma_1},\ldots,y_rx^{\sigma_r}\right),$$

where $F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}$ is generalized multiple hypergeometric function of Srivastava and Daoust ([11],[12],[13]) and $\sigma_i>0$, $i=1,\ldots,r$;

we have

$$(3.2.3)\ \frac{(1+v)^{\alpha}}{1-\beta v}F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\ [(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(c):\psi',\ldots,\psi^{(r)}],[(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix}\right.$$

$$y_1 x^{\sigma_1}(1+v)^{\sigma_1},\ldots y_r x^{\sigma_r}(1+v)^{\sigma_r} \Bigg)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}T^n_{\alpha+\beta n}\left\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix}\right.\right.$$

$$y_1 x^{\sigma_1},\ldots y_r x^{\sigma_r} \Bigg)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\ldots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^r)_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\ldots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^r)_{m_r\delta_j^{(r)}}}$$

$$y_1^{m_1}\ldots y_r^{m_r}T^n_{\alpha+\beta n}\left\{x^{m_1\sigma_1}+\ldots+x^{m_r\sigma_r}\right\},$$

where $v=xz(1+v)^{\beta+1}$

Now making an appeal to (3.2.2) and

(3.2.4) $T_k^n\left\{x^{\gamma}\right\}=(k+\gamma)_n x^{\gamma+n}$,

we obtain

(3.2.5) $$\frac{(1+v)^{\alpha}}{1-\beta v}F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],[(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(c):\psi',\ldots,\psi^{(r)}][(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix}\right.$$

$$y_1 x^{\sigma_1}(1+v)^{\sigma_1},\ldots,y_r x^{\sigma_r}(1+v)^{\sigma_r} \Bigg)$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\ldots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^r)_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\ldots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^r)_{m_r\delta_j^{(r)}}}$$

$$\frac{(\alpha+\beta n)_n (\alpha+(\beta+1)n)_{m_1\sigma_1+...+m_r\sigma_r}}{(\alpha+\beta n)_{m_1\sigma_1+...+m_r\sigma_r}} y_1^{m_1}...y_r^{m_r} x^{n+m_1\sigma_1+...+m_r\sigma_r}$$

$$= \sum_{n=0}^{\infty} \frac{(zx)^n}{n!} (\alpha+\beta n)_n F_{C+1:D';...;D^{(r)}}^{A+1:B';...;B^{(r)}} \left(\begin{matrix} [(a):\theta',...,\theta^{(r)}], [\alpha+(\beta+1)n:\sigma_1,....,\sigma_r]: \\ [(c):\psi',...,\psi^{(r)}], [\alpha+\beta n:\sigma_1,....,\sigma_r]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];....;[(b^r):\phi^r]; \\ [(d'):\delta'];....;[(d^r):\delta^r]; \end{matrix} y_1 x^{\sigma_1},....,y_r x^{\sigma_r} \right) .$$

Now replacing z by t/x and $y_i\ x^{\sigma_i}$ by z_i $(i=1,..,r)$, we finally derive the main generating relation (3.2.1).

3.3. Special Case. When $\beta=0$. In this case $v=t(1+v)$. Therefore generating relation (3.2.1) reduces to

$$(3.3.1)\ \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha)_n F_{C+1:D';...;D^{(r)}}^{A+1:B';...;B^{(r)}} \left(\begin{matrix} [(a):\theta',...,\theta^{(r)}], [\alpha+n:\sigma_1,...,\sigma_r]: \\ [(c):\psi',...,\psi^{(r)}], [\alpha:\sigma_1,...,\sigma_r]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];....;[(b^r):\phi^r]; \\ [(d'):\delta'];....;[(d^r):\delta^r]; \end{matrix} z_1,...z_r \right) ,$$

$$= (1-t)^{-\alpha} F_{C:D';...;D^{(r)}}^{A:B';...;B^{(r)}} \left(\begin{matrix} [(a):\theta',...,\theta^{(r)}], [(b'):\phi'];....;[(b^r):\phi^r]; \\ [(c):\psi',...,\psi^{(r)}], [(d'):\delta'];....;[(d^r):\delta^r]: \end{matrix} \right.$$

$$\left. z_1(1-t)^{-\sigma_1},...,z_r(1-t)^{-\sigma_r} \right) ,$$

provided that $\sigma_i > 0$, $|z_i|<1$, $1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0$, $i=1,...,r$.

3.4. Special Case. When β=-1. In this case $v=t$. Therefore

generating relation (3.2.1) reduces to

$$(3.4.1)\quad \sum_{n=0}^{\infty}\binom{\alpha}{n}t^n F^{A+1:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\ [\alpha+1:\sigma_1,\ldots,\sigma_r]:\\ [(c):\psi',\ldots,\psi^{(r)}],[\alpha+1-n:\sigma_1,\ldots,\sigma_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix} z_1,\ldots z_r\right),$$

$$=(1+t)^{\alpha}F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^r):\phi^r];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^r):\delta^r];\end{matrix}\right.$$

$$\left. z_1(1+t)^{\sigma_1},\ldots,z_r(1+t)^{\sigma_r}\right),$$

valid if $|z_i|<1$, $\sigma_i>0$, and $1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,\quad i=1,\ldots,r.$

3.5 Special Case. When $\sigma_1=\ldots=\sigma_r=1$ and specializing other paramters, (3.2.1) gives :

$$(3.5.1)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n F_D^{(r)}(\alpha+(\beta+1)n,b_1,\ldots,b_r;\alpha+\beta n;z_1,\ldots,z_r)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\prod_{i=1}^{r}[1-z_i(1+v)]^{-b_i},\qquad |z_i|<1,\ i=1,\ldots,r$$

and

$$(3.5.2)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n\phi_D^{(r)}(\alpha+(\beta+1)n,b_1,\ldots,b_{r-1}-;\alpha+\beta n;z_1,\ldots,z_r)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\prod_{i=1}^{r-1}[1-z_i(1+v)]^{-b_i}\ e^{z_r(1+v)},\qquad |z_i|<1,\ i=1,\ldots,r.$$

(3.3.1) gives :

(3.5.3) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F_D^{(r)}(\alpha+n,b_1,\ldots,b_r;\alpha;z_1,\ldots,z_r)$$

$$=(1-t)^{-\alpha}\prod_{i=1}^{r}[1-z_i/(1-t)]^{-b_i}\quad, \qquad |z_i|<1,\ i=1,\ldots,r$$

and

(3.5.4) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n \phi_D^{(r)}(\alpha+n,b_1,\ldots,b_{r-1},-;\alpha;z_1,\ldots,z_r)$$

$$=(1-t)^{-\alpha}\prod_{i=1}^{r-1}[1-z_i/(1-t)]^{-b_i}.\ e^{z_r/(1-t)}\quad, \qquad |z_i|<1,\ i=1,\ldots,r;$$

while (3.4.1) gives

(3.5.5) $$\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n F_D^{(r)}(\alpha+1,b_1,\ldots,b_r;\alpha+1-n;z_1,\ldots,z_r)$$

$$=(1+t)^{\alpha}\prod_{i=1}^{r}[1-z_i(1+t)]^{-b_i}\quad, \qquad |z_i|<1,\ i=1,\ldots,r$$

and

(3.5.6) $$\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n \phi_D^{(r)}(\alpha+1,b_1,\ldots,b_{r-1},-;\alpha+1-n;z_1,\ldots,z_r)$$

$$=(1+t)^{\alpha}\prod_{i=1}^{r-1}[1-z_i(1+t)]^{-b_i}\ e^{z_r(1+t)}\quad, \qquad |z_i|<1,\ i=1,\ldots,r;$$

where $v=t(1+v)^{\beta+1}$, $F_D^{(r)}$ is fourth multiple hypergeometric function due to Lauricella [8] while $\phi_D^{(r)}$ is its confluent form.

3.6. Other Special Cases.

Case (a) When r=3, $\sigma_1=\sigma_2=\sigma_3=1$ and specializing other paramters, the results (3.2.1), (3.3.1) and (3.4.1) reduce respectively to

$$(3.6.1)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+n\beta)_n F^{(3)}\left[\begin{matrix}(a),\alpha+(\beta+1)n:(b);(b');(b''):(c);(c');(c'');\\(f),\alpha+\beta n:(g);(g');(g''):(h);(h');(h'');\end{matrix}\, z_1,z_2,z_3\right]$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}F^{(3)}\left[\begin{matrix}(a):(b);(b');(b''):(c);(c');(c'');\\(f):(g);(g'):(g''):(h);(h');(h'');\end{matrix}\, z_1(1+v),z_2(1+v),z_3(1+v)\right],$$

where $v=t(1+v)^{\beta+1}$ and $F^{(3)}$ is generalized hypergeometric series of three variables due to Srivastva [10].

$$(3.6.2)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F^{(3)}\left[\begin{matrix}(a),\alpha+n:(b);(b');(b''):(c);(c');(c'');\\(f),\alpha:(g);(g');(g''):(h);(h');(h'');\end{matrix}\, z_1,z_2,z_3\right]$$

$$=(1-t)^{-\alpha}F^{(3)}\left(\begin{matrix}(a):(b);(b');(b''):(c);(c');(c'');\\(f):(g);(g'):(g''):(h);(h');(h'');\end{matrix}\, \frac{z_1}{(1-t)},\frac{z_2}{(1-t)},\frac{z_3}{(1-t)}\right).$$

$$(3.6.3)\quad \sum_{n=0}^{\infty}t^n\binom{\alpha}{n}F^{(3)}\left(\begin{matrix}(a),\alpha+1:(b);(b');(b''):(c);(c');(c'');\\(f),\alpha+1-n:(g);(g');(g''):(h);(h');(h'');\end{matrix}\, z_1,z_2,z_3\right)$$

$$=(1+t)^{\alpha}F^{(3)}\left(\begin{matrix}(a):(b);(b');(b''):(c);(c');(c'');\\(f):(g);(g');(g''):(h);(h');(h'');\end{matrix}\, z_1(1+t),z_2(1+t),z_3(1+t)\right).$$

Case (b) When r=2, $\sigma_1=\sigma_2=1$ and specializing other parameters, we derive from (3.2.1)

$$(3.6.4)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n F_1(\alpha+(\beta+1)n,b_1,b_2;\alpha+\beta n;z_1,z_2)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}[1-z_1(1+v)]^{-b_1}[1-z_2(1+v)]^{-b_2},\quad v=t(1+v)^{\beta+1},$$

where F_1 is Appell function of two variables.

(3.6.5) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha+\beta n)_n\,\phi_1(\alpha+(\beta+1)n,b_1;\alpha+\beta n;z_1,z_2)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\left[1-z_1(1+v)\right]^{-b_1}e^{z_2(1+v)},\qquad v=t(1+v)^{\beta+1},$$

where ϕ_1 is confluent series of Appell's series F_1.

From (3.3.1), we derive

(3.6.6) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n F_1(\alpha+n,b_1,b_2;\alpha;z_1,z_2)$$

$$=(1-t)^{-\alpha}\left[1-z_1/(1-t)\right]^{-b_1}\left[1-z_2/(1-t)\right]^{-b_2}$$

and

(3.6.7) $$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\alpha)_n\phi_1(\alpha+n,b_1;\alpha;z_1,z_2)=(1-t)^{-\alpha}\left[1-\frac{z_1}{1-t}\right]^{-b_1}e^{z_2/1-t}.$$

Further from (3.4.1), we deduce

(3.6.8) $$\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n F_1(\alpha+1,b_1,b_2;\alpha+1-n;z_1,z_2)$$

$$=(1+t)^{\alpha}\left[1-z_1(1+t)\right]^{-b_1}\left[1-z_2(1+t)\right]^{-b_2}$$

and

(3.6.9) $$\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n\phi_1(\alpha+1,b_1;\alpha+1-n;z_1,z_2)$$

$$=(1+t)^{\alpha}\left[1-z_1(1+t)\right]^{-b_1}e^{z_2(1+t)}.$$

Case (c) When $r=1$ and $\sigma_1=1$ the generating relations (3.2.1), (3.3.1) and (3.4.1) reduce respectively to

$$(3.6.10)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha+\beta n)_n \; {}_1F_1\left[\begin{matrix} \alpha+(\beta+1)n; \\ \alpha+\beta n; \end{matrix} \; z\right] = \frac{(1+v)^{\alpha}}{1-\beta v} e^{(1+v)z},$$

where $v = t(1+v)^{\beta+1}$,

$$(3.6.11)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha)_n \; {}_1F_1\left[\begin{matrix} \alpha+n; \\ \alpha; \end{matrix} \; z\right] = (1-t)^{-\alpha} e^{-z(1-t)}$$

and

$$(3.6.12)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \binom{\alpha}{n} t^n \; {}_1F_1\left[\begin{matrix} \alpha+1; \\ \alpha+1-n; \end{matrix} \; z\right] = (1+t)^{\alpha} e^{z(1+t)}.$$

3.7 Some more Generating Functions of other Multiple Hypergeometric Functions. In this section, we shall derive generating relations of multiple hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ with their confluent forms $\phi_2^{(r)}$, $\psi_2^{(r)}$ of Lauricella [8], ${}^{(k)}_{(1)}E_D^{(r)}$, ${}^{(k)}_{(2)}E_D^{(r)}$ of Exton ([4],[5]), ${}^{(k)}_{(1)}E_C^{(r)}$ of Chandel [1], ${}^{(k)}F_{AC}^{(r)}$, ${}^{(k)}F_{AD}^{(r)}$, ${}^{(k)}F_{BD}^{(r)}$ and their confluent forms ${}^{(k)}_{(1)}\phi_{AC}^{(r)}$, ${}^{(k)}_{(2)}\phi_{AC}^{(r)}$, ${}^{(k)}_{(1)}\phi_{AD}^{(r)}$, ${}^{(k)}_{(1)}\phi_{BD}^{(r)}$, ${}^{(k)}_{(2)}\phi_{BD}^{(r)}$ of Chandel and Gupta [2], ${}^{(k)}F_{CD}^{(r)}$ of Karlsson [7] and for its confluent forms ${}^{(k)}_{(1)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(2)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(3)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(4)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(5)}\phi_{CD}^{(r)}$, ${}^{(k)}_{(6)}\phi_{CD}^{(r)}$ due to Chandel and Vishwakarma [3] and also for more confluent forms ${}^{(k)}_{(2)}\phi_{AD}^{(r)}$, ${}^{(k)}_{(3)}\phi_{BD}^{(r)}$, ${}^{(k)}_{(1)}\phi_{D}^{(r)}$, ${}^{(k)}_{(2)}\phi_{D}^{(r)}$, ${}^{(k)}_{(1)}\phi_{C}^{(r)}$ due to Vishwakarma [19].

Choosing $\sigma_1 = \ldots = \sigma_r = 1$ and specializing other parameters in (3.2.1), we derive the following generating relations :

$$(3.7.1)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha+n\beta)_n F_{1:1;\ldots;1}^{2:1,\ldots;1}\left(\begin{matrix} [\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:[b_1:1];\ldots;[b_r:1]; \\ [\alpha+n\beta:1,\ldots,1]:[c_1:1];\ldots;[c_r:1]; \end{matrix} \; z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} F_A^{(r)}(a, b_1, \ldots, b_r; c_1, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)),$$

$$\sum_{i=1}^{r} |(1+v)z_i| < 1 .$$

$$(3.7.2)\ \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha+n\beta)_n F_{2:-;\ldots;-}^{1:2,\ldots;2}\left(\begin{matrix} [\alpha+(\beta+1)n:1,\ldots,1]:[a_1:1],[b_1:1];\ldots;[a_r,1],[b_r:1]; \\ [\alpha+n\beta:1,\ldots,1]:[c:1,\ldots,1]:-;\ldots;-; \end{matrix}\ z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} F_B^{(r)}(a_1, \ldots, a_r, b_1, \ldots, b_r; c; z_1(1+v), \ldots, z_r(1+v)),$$

$$|(1+v)z_i| < 1,\ i = 1, \ldots, r .$$

$$(3.7.3) \sum_{n=0}^{\infty} \frac{(\alpha+n\beta)_n}{n!} t^n F_{1:1;\ldots;1}^{3:-;\ldots;-}\left(\begin{matrix} [\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],[b:1,\ldots,1]:-;\ldots;-: \\ [\alpha+n\beta:1,\ldots,1]:[c_1:1];\ldots;[c_r:1]; \end{matrix}\ z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} F_C^{(r)}(a, b; c_1, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)),$$

$$\sum_{i=1}^{r} [|(1+v)z_i|]^{1/2} < 1 .$$

$$(3.7.4) \quad \sum_{n=0}^{\infty} \frac{(\alpha+n\beta)_n}{n!} t^n F_{1:1;\ldots;1}^{2:-;\ldots;-}\left(\begin{matrix} [\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:-;\ldots;-; \\ [\alpha+n\beta:1,\ldots,1]:[c_1:1];\ldots;[c_r:1]; \end{matrix}\ z_1,\ldots,z_r\right)$$

$$= \frac{(1+v)^{\alpha}}{1-\beta v} \psi_2^{(r)}(a; c_1, \ldots, c_r; z_1(1+v), \ldots, z_r(1+v)),$$

$$(3.7.5) \quad \sum_{n=0}^{\infty} \frac{(\alpha+n\beta)_n}{n!} t^n F_{2:-;\ldots;-}^{1:1,\ldots;1}\left(\begin{matrix} [\alpha+(\beta+1)n:1,\ldots,1]:[b_1:1];\ldots;[b_r:1]; \\ [\alpha+n\beta:1,\ldots,1],[c:1,\ldots,1]:-;\ldots;-; \end{matrix}\ z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\phi_2^{(r)}(b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.6)\sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{2:1,\ldots;1}_{3:-;\ldots;-}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]: \\ [\alpha+n\beta:1,\ldots,1]:[c:1,\ldots,1,-,\ldots,-],[c':-,\ldots,-,1,\ldots,1]:\end{array}\right.$$

$$\left.\begin{array}{l}[b_1:1];\ldots;[b_r:1]; \\ -;\ldots;-;\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}E_D^{(r)}(a,b_1,\ldots,b_r;c,c';z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.7)\sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{3:1,\ldots;1}_{2:-;\ldots;-}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1,-,\ldots,-],[a':-,\ldots,-,1,\ldots,1]: \\ [\alpha+n\beta:1,\ldots,1]:[c:1,\ldots,1]:\end{array}\right.$$

$$\left.\begin{array}{l}b_1,\ldots b_r; \\ -;\ldots;-;\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}E_D^{(r)}(a,a',b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)).$$

$$(3.7.8)\quad\sum_{n=0}^{\infty}\frac{(\alpha+n\beta)_n}{n!}t^n F^{4:-;\ldots;-}_{1:1;\ldots;1}\left(\begin{array}{l}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1,-,\ldots,-], \\ [\alpha+n\beta:1,\ldots,1]:\end{array}\right.$$

$$\left.\begin{array}{l}[a':-,\ldots,-,1,\ldots,1],[b:1,\ldots,1]:-;\ldots;-; \\ [c_1:1];\ldots;[c_r:1];;\end{array} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}E_C^{(r)}(a,a',b;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v))$$

where ${}^{(k)}_{(1)}E_D^{(r)}$, ${}^{(k)}_{(2)}E_D^{(r)}$ are multiple hypergeometric functions of Exton ([4],[5]) related to Lauricella's $F_D^{(r)}$ while ${}^{(k)}_{(1)}E_C^{(r)}$ is multiple hypergeometric function of Chandel [1] related to Lauricella's $F_C^{(r)}$.

$$(3.7.9)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n t^n}{n!}F^{3:-;\ldots;-;1,\ldots,1}_{1:1,\ldots,1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],\\ [\alpha+\beta n:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b:1,\ldots,1,-,\ldots,-,]:-;\ldots;-;[b_{k+1}:1];\ldots;[b_r:1];\\ [c_1:1];\ldots;[c_r:1];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}{}^{(k)}F_{AC}^{(r)}(a,b,b_{k+1},\ldots,b_r;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.10)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n t^n}{n!}F^{2:1;\ldots;1}_{2:-;\ldots;-;1,\ldots,1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-,\ldots,-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1];\\ -;\ldots;-;[c_{k+1}:1];\ldots;[c_r:1];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}{}^{(k)}F_{AD}^{(r)}(a,b_1,\ldots,b_r;c,c_{k+1},\ldots,c_n;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.11)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n t^n}{n!}F^{2:1;\ldots;1;2;\ldots;2}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1,-,\ldots,-]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];[a_{k+1}:1],[b_{k+1}:1];\ldots;[a_n:1],[b_n:1];\\ -;\ldots;-;\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}F_{BD}^{(r)}\left(a,a_{k+1},\ldots,a_r,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right).$$

$$(3.7.12)\sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n t^n}{n!}F_{2:-;\ldots;-;1;\ldots;1}^{3:1;\ldots;1;-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],[b:-;\ldots;-;1,\ldots,1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];-;\ldots;-;\\ -;\ldots;-;[c_{k+1}:1];\ldots;[c_r:1];\end{matrix}z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}F_{CD}^{(r)}\left(a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.13)\sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F_{1:1;\ldots;1}^{3:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],[b:1,\ldots,1,-,\ldots,-]:-;\ldots;-;\\ [\alpha+\beta n:1,\ldots,1],[c_1:1];\ldots;[c_r:1];\end{matrix}z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi_{AC}^{(r)}\left(a,b;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.14)\sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F_{1:1;\ldots;1}^{2;-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1],-;\ldots;-;[b_{k+1}:1];\ldots;[b_r:1];\\ [\alpha+\beta n:1,\ldots,1],[c_1:1];\ldots;[c_r:1];\end{matrix}z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}\phi_{AC}^{(r)}\left(a,b_{k+1},\ldots,b_r;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.15)\sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F_{2:-;\ldots;-}^{2:1;\ldots;1}\left(\begin{matrix}[\alpha(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:[b_1:1];\ldots;[b_r:1];\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:-;\ldots;-;\end{matrix}z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi_{AD}^{(r)}\left(a,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)\right)$$

$$(3.7.16)\ \frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1,\ldots,1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots,1]:[b_1:1];\ldots;[b_r:1]\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:-;\ldots;-;\end{matrix} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^\alpha}{1-\beta v}\,{}^{(k)}_{(1)}\phi^{(r)}_{BD}(a,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.17)\ \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!} t^n F^{1:1;\ldots;1;2;\ldots;2}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha(\beta+1)n:1,\ldots,1]:[b_1:1];\ldots;[b_k:1];\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[a_{k+1}:1],[b_{k+1}:1];\ldots;[a_r:1],[b_r:1];\\ -;\ldots;-;\end{matrix} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^\alpha}{1-\beta v}\,{}^{(k)}_{(2)}\phi^{(r)}_{BD}(a_{k+1},\ldots,a_r,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.18)\ \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!} t^n F^{3:-;\ldots;-}_{2:-;\ldots;-;1,\ldots,1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1]:[a:1,\ldots,1],[b:-;\ldots;-,1,\ldots,1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-;\ldots;-]:-;\ldots;-;\end{matrix}\right.$$

$$\left.\begin{matrix}-;\ldots;-;\\ [c_{k+1}:1];\ldots;[c_r:1];\end{matrix} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^\alpha}{1-\beta v}\,{}^{(k)}_{(1)}\phi^{(r)}_{CD}(a,b;c,c_{k+1},\ldots,c_n;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.19)\ \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!} t^n F^{2:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-1,\ldots,-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];-;\ldots;-;\\ -;\ldots;-;[b_{k+1}:1];\ldots;[b_r:1];\end{matrix} z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}\phi^{(r)}_{CD}\left(a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v_r)\right),$$

$$(3.7.20)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[b:-,\ldots,-,1,\ldots,1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-,\ldots,-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];-;\ldots;-;\\ -;\ldots;-;[c_{k+1}:1];\ldots;[c_r:1];\end{matrix}\,z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(3)}\phi^{(r)}_{CD}\left(b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.21)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{3:1;\ldots;1;-;\ldots;-}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots1]:\\ [\alpha+\beta n:1,\ldots,1],[c:1,\ldots,1,-,\ldots,-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b:-,\ldots,-,1,\ldots,1]:[b_1:1];\ldots;[b_k:1];-;\ldots;-;\\ -;\ldots;-;\end{matrix}\,z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(4)}\phi^{(r)}_{CD}\left(a,b,b_1,\ldots,b_k;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.22)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{3:1;\ldots;1;-;\ldots;-}_{1:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots1]:\\ [\alpha+\beta n:1,\ldots,1]:-;\ldots;-;\end{matrix}\right.$$

$$\left.\begin{matrix}[b:-,\ldots,-,1,\ldots,1]:[b_1:1];\ldots;[b_k:1];-;\ldots;-;\\ [c_{k+1}:1];\ldots;[c_r:1];\end{matrix}\,z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(5)}\phi^{(r)}_{CD}\left(a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.23)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1;-;\ldots;-}_{1:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1]: \\ [\alpha+\beta n:1,\ldots,1]:-;\ldots;-;\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];-;\ldots;-; \\ [c_{k+1}:1];\ldots;[c_r:1]\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(6)}\phi^{(r)}_{CD}\left(a,b_1,\ldots,b_k;c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.24)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1}_{1:-;\ldots;-;1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1]: \\ [\alpha+\beta n:1,\ldots,1]:-;\ldots;-;\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1]; \\ [c_{k+1}:1];\ldots;[c_r:1];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}\phi^{(r)}_{AD}\left(a,b_1,\ldots,b_r;c_{k+1},\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.25)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1;2;\ldots;2}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1,-;\ldots;-]: \\ [\alpha+\beta n:1,\ldots,1]:[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_k:1];[a_{k+1}:1],[b_{k+1}:1];\ldots;[a_r:1],[b_r:1]; \\ -;\ldots;-;\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(3)}\phi^{(r)}_{BD}\left(a,a_{k+1},\ldots,a_r,b_1,\ldots,b_k;c;z_1(1+v),\ldots,z_r(1+v)\right),$$

$$(3.7.26)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}t^n F^{2:1;\ldots;1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1]: \\ [\alpha+\beta n:1,\ldots,1]:[c:1,\ldots,1,-;\ldots;-]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1]; \\ -;\ldots;-;\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi_D^{(r)}(a,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.27)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}\,t^n F^{2:1;\ldots;1}_{2:-;\ldots;-}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1,-,\ldots,-]: \\ [\alpha+\beta n:1,\ldots,1]:[c:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b_1:1];\ldots;[b_r:1]; \\ -;\ldots;-;\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(2)}\phi_D^{(r)}(a,b_1,\ldots,b_r;c;z_1(1+v),\ldots,z_r(1+v)),$$

$$(3.7.28)\quad \sum_{n=0}^{\infty}\frac{(\alpha+\beta n)_n}{n!}\,t^n F^{3:-;\ldots;-}_{1:1;\ldots;1}\left(\begin{matrix}[\alpha+(\beta+1)n:1,\ldots,1],[a:1,\ldots 1,-;\ldots;-]: \\ [\alpha+\beta n:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[b:1,\ldots,1]:-;\ldots;-; \\ [c_1:1];\ldots;[c_r:1];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{(1+v)^{\alpha}}{1-\beta v}\,{}^{(k)}_{(1)}\phi_C^{(r)}(a,b;c_1,\ldots,c_r;z_1(1+v),\ldots,z_r(1+v)).$$

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ON SOME ASSOCIATED POLYNOMIALS DEFINED THROUGH GENERATING FUNCTIONS

CHAPTER -IV

ON SOME ASSOCIATED POLYNOMIALS DEFINED THROUGH GENERATING FUNCTIONS

4.1 Intoduction. Gould [6] considered a class of generalized Humbert polynomials defined by

$$(4.1.1)\quad \left(C - mxt + yt^{m}\right)^{p} = \sum_{n=0}^{\infty} P_n(m,x,y,p,C)t^{n},$$

where m is positive integer and other parameters are unrestricted in general.

Panda [7] introduced polynomials $\{g_n^c(x,r,s)/ n = 0,1 \ldots.\}$ generated by

$$(4.1.2)\quad (1-t)^{-C} G\left(\frac{xt^{s}}{(1-t)^{r}}\right) = \sum_{n=0}^{\infty} g_n^c(x,r,s)t^{n},$$

where c is arbitrary, r is any integer positive or negative and s=1,2,3,...

On the other hand Srivastava [9] considered a class of generalized Hermite polynomials defined by the generating function

$$(4.1.3)\quad \sum_{n=0}^{\infty} \gamma_m^{(x)} \frac{t^{n}}{n!} = G\left(mxt - t^{m}\right),$$

where and in (4.1.2)

$$(4.1.4)\quad G(z) = \sum_{n=0}^{\infty} \gamma_n z^{n}, \qquad \gamma_0 \neq 0 .$$

To unify the study of the three general classes of polynomials defined through (4.1.1), (4.1.2) and (4.1.3), Chandel [1] considered a generating function

$$(4.1.5)\ \left(C-mxt+yt^m\right)^p G\left[\frac{r^r x t^s}{\left(C-mxt+yt^m\right)^r}\right]=\sum_{n=0}^{\infty} R_n^p(m,x,y,r,s,C)t^n,$$

where m,s are positive integers, other parameters are unrestricted in general and $G(z)$ is given by (4.1.4). He also discussed the interesting special case when $\gamma_n=\frac{(-1)^n}{n!}$.

$$(4.1.6)\ \left(C-mxt+yt^m\right)^p \exp\left[-\frac{xr^r t^s}{\left(C-mxt+yt^m\right)^r}\right]=\sum_{n=0}^{\infty} C_n^p(m,x,y,r,s,C)t^n.$$

Further to study associated polynomials, Chandel and Bhargava [4] considered the special case of (4.1.5) for $\gamma_n=\frac{(q)_n}{n!}$, $s=r$, $m=2$

$$(4.1.7)\ \left(C-2xt+yt^2\right)^p \exp\left[1-\frac{xr^r t^r}{\left(C-2xt+yt^2\right)^r}\right]^{-q}=\sum_{n=0}^{\infty} B_n^{(p,q)}(2,x,y,r,s,C)t^n.$$

Prompted by (4.1.5), Chandel and Dwivedi [3] considered another generating function :

$$(4.1.8)\ \left(C-mxt+yt^m\right)^p G\left[\frac{yr^r t^s}{\left(C-mxt+yt^m\right)^r}\right]=\sum_{n=0}^{\infty} R_n^p(m,x,y,r,s,C)t^n$$

different from (4.1.5). He also considered their special cases when $\gamma_n=\frac{(-1)^n}{n!}$.

$$(4.1.9)\ \left(C-mxt+yt^m\right)^p \exp\left[-\frac{r^r y t^s}{\left(C-mxt+yt^m\right)^r}\right]=\sum_{n=0}^{\infty} E_n^p(m,x,y,r,s,C)t^n,$$

and when $\gamma_n = \frac{(q)_n}{n!}$

(4.1.10) $$\left(C-mxt+yt^m\right)^p\left[1-\frac{r^r yt^s}{\left(C-mxt+yt^m\right)^r}\right]^{-q} = \sum_{n=0}^{\infty} F_n^{(p,q)}(m,x,y,r,s,C)t^n,$$

but in no case he tried to introduce associated polynomials.

In this Chapter, we shall introduce associated polynomials for the polynomials defined by (4.1.10).

Further Dwivedi [5] considered another generating function

(4.1.11) $$\left(C-mxt+yt^m\right)^p G\left(\frac{r^r zt^s}{\left(C-mxt+yt^m\right)^r}\right) = \sum_{n=0}^{\infty} C_n^p(m,x,y,z,t,r,C)t^n$$

with z independent of x and y. He also considered its special cases for

$\gamma_n = \frac{(-1)^n}{n!}$ and $\gamma_n = \frac{(q)_n}{n!}$.

To study associated polynomials of class (4.1.11) Chandel and Dwivedi [2] considered the case of $\gamma_n = \frac{(-1)^n}{n!}$, $s=r$ and $m=2$

(4.1.12) $$\left(C-2xt+yt^2\right)^p \exp\left(-\frac{r^r zt^r}{\left(C-2xt+yt^2\right)^r}\right) = \sum_{n=0}^{\infty} E_n^p(2,x,y,z,t,r,C)t^n.$$

Now in the present Chapter, we shall also introduce polynomials related to the polynomials for which $\gamma_n = \frac{(q)_n}{n!}$, $s=r$ and $m=2$.

(4.1.13) $$\left(C-2xt+yt^2\right)^p\left(1-\frac{r^r zt^r}{\left(C-2xt+yt^2\right)^r}\right)^{-q} = \sum_{n=0}^{\infty} M_n^{(p,q)}(2,x,y,z,r,s,C)t^n$$

4.2. The polynomials $\{A_n^{p,q}(x,y,r,C)/n=0,1,2....\}$. For $s=r$, $m=2$, (4.1.10) defines

$$(4.2.1)\ (C-2xt+yt^2)^p\left(1-\frac{r^r yt^r}{(C-2xt+yt^2)^r}\right)^{-q}=\sum_{n=0}^{\infty}F_n^{(p,q)}(2,x,y,r,s,C)t^n\ .$$

Consider

$$(4.2.2)\ \sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,r,C)F_{n-k}^{(p-k,q)}(2,x,y,r,r,C)=0\ ,$$

and

$$(4.2.3)\ A_0^{(p,q)}(x,y,r,C)=1.$$

Therefore

$$1=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}A_k^{(p,q)}(x,y,r,C)F_{n-k}^{(p-k,q)}(2,x,y,r,r,C)$$

$$=\sum_{n=0}^{\infty}\ \sum_{k=0}^{n}A_k^{(p,q)}(x,y,r,C)F_n^{(p-k,q)}(2,x,y,r,r,C)t^{n+k}$$

$$=\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,r,C)t^k\sum_{n=0}^{\infty}F_n^{(p-k,q)}(2,x,y,r,r,C)t^n$$

$$=\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,r,C)t^k(C-2xt+yt^2)^{p-q}\left[1-\frac{r^r yt^r}{(C-2xt+yt^2)^r}\right]^{-q}$$

$$=(C-2xt+yt^2)^p\left[1-\frac{r^r yt^r}{(C-2xt+yt^2)^r}\right]^{-q}\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,r,C)\frac{t^k}{(C-2xt+yt^2)^k}$$

Thus

$$(4.2.4)\ (C-2xt+yt^2)^{-p}\left[1-\frac{r^r yt^r}{(C-2xt+yt^2)^r}\right]^q$$

$$=\sum_{k=0}^{\infty} A_k^{(p,q)}(x,y,r,C)\left[\frac{t}{C-2xt+yt^2}\right]^k .$$

Take $t/(C-2xt+yt^2)=w$

Therefore

$$(4.2.5)\ \sum_{n=0}^{\infty} A_n^{(p,q)}(x,y,r,C)w^n=\left(\frac{w}{t}\right)^p\left[1-r^r yw^r\right]^q$$

where

$$(4.2.6)\ t=\frac{2xw+1+\sqrt{(2xw+1)^2-4ycw^2}}{2xy} .$$

Thus

$$(4.2.7)\ \sum_{k=0}^{\infty} A_k^{(p,q)}(x,y,r,C)w^n=\left[\frac{2xw+1+\sqrt{(2xw+1)^2-4ycw^2}}{2yw^2}\right]^{-p}\left[1-r^r yw^r\right]^q$$

4.3. Applications of Generating Relation. An appeal to generating relation (4.2.7) gives

$$(4.3.1)\ A_n^{(p+p',q+q')}(x,y,r,C)=\sum_{k=0}^{n} A_k^{(p,q)}(x,y,r,C)A_{n-k}^{(p',q')}(x,y,r,C),$$

which can be generalized in the following form :

$$(4.3.2)\ A_n^{(p_1+\ldots+p_m;q_1,\ldots,q_m)}(x,y,r,C)=\sum_{i_1+\ldots+i_m=n} A_{i_1}^{(p,q)}(x,y,r,C)A_{i_m}^{(p_m,q_m)}(x,y,r,C).$$

An appeal to (2.7), further shows that

(4.3.3) $A_n^{(p,q+1)}(x,y,r,C)=A_n^{(p,q)}(x,y,r,C)-r^r y A_{n-r}^{(p,q)}(x,y,r,C)$

(4.3.4) $A_n^{(p,q)}(x,y,r,C)=\sum_{k=0}^{[n/r]} A_{n-rk}^{(p,q)}(x,y,r,C)(r^r y)^k,$

(4.3.5) $A_n^{(p,q-q')}(x,y,r,C)=\sum_{k=0}^{n} A_{n-rk}^{(p,q)}(x,y,r,C)\frac{(q')_k (r^r y)^k}{k!},$

(4.3.6) $A_n^{(p,q)}(x,y,r,C)=\sum_{k=0}^{n} A_{n-k}^{(p',q')}(x,y,r,C)A_k^{(p-p',q-q')}(x,y,r,C),$

and

(4.3.7) $A_n^{p,q}(x,y,r,C)=\sum_{k=0}^{n} A_{n-k}^{(p-p',q')}(x,y,r,C)A_k^{(p,q-q')}(x,y,r,C).$

4.4. The polynomials $\{N_n^{(a,b;c,d)}(x,y,r,C)/n=0,1,2,...\}$.

Consider

(4.4.1) $N_n^{(a,b;c,d)}(x,y,r,C)=\sum_{k=0}^{n} A_k^{(a,b)}(x,y,r,C)F_{n-k}^{(c-k,d)}(2,x,y,r,r,C),$

Therefore

$$\sum_{n=0}^{\infty} N_n^{(a,b;c,d)}(x,y,r,C)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_k^{(a,b)}(x,y,r,C)F_{n-k}^{(c-k,d)}(2,x,y,r,r,C)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_k^{(a,b)}(x,y,r,C)t^k F_n^{(c-k,d)}(2,x,y,r,r,C)t^n$$

$$=\sum_{k=0}^{\infty} A_k^{(a,b)}(x,y,r,C)t^k\left(C-2xt+yt^2\right)^{c-k}\left[1-\frac{r^r y t^r}{\left(C-2xt+yt^2\right)^r}\right]^{-d}$$

$$=\left[1-\frac{r^r y t^r}{\left(C-2xt+yt^r\right)^r}\right]^{-d}\left(C-2xt+yt^2\right)^c\sum_{k=0}^{\infty}A_k^{(a,b)}(x,y,r,C)\left[\frac{t}{\left(C-2xt+yt^2\right)}\right]^k .$$

(By (2.3)) .

Hence

$$(4.4.2)\ \sum_{n=0}^{\infty}N_n^{(a,b;c,d)}(x,y,r,C)t^n$$

$$=\left(C-2xt+yt^2\right)^{c-a}\left[1-\frac{r^r y t^r}{\left(C-2xt+yt^2\right)^r}\right]^{-(d-b)}$$

$$=\sum_{n=0}^{\infty}F_n^{(c-a,d-b)}(2,x,y,r,r,C)t^n .$$

Thus

$$(4.4.3)\ N_n^{(a,b;c,d)}(x,y,r,C)=F_n^{(c-a,d-b)}(2,x,y,r,r,C).$$

For $a=c$, (4.4.2) gives

$$(4.4.4)\ N_n^{(a,b;a,d)}(x,y,r,C)t^n=\left[1-\frac{r^r y t^r}{\left(C-2xt+yt^2\right)^r}\right]^{b-d} .$$

For $b=d$, (4.4.2), gives

$$(4.4.5)\ \sum_{n=0}^{\infty}N_n^{(a,b;c,d)}(x,y,r,C)t^n=\left(C-2xt+yt^2\right)^{c-a} .$$

For $a=c$, $b=d$.

$$(4.4.6)\ \sum_{n=0}^{\infty}N_n^{(a,b;a,b)}(x,y,r,C)t^n=1 .$$

By an appeal to (4.2), we have

$$\sum_{n=0}^{\infty} N_n^{(a+a',b+b';c+c',d+d')}(x,y,r,C)t^n$$

$$=\left(C-2xt+yt^2\right)^{c+c'-(a+a')}\left[1-\frac{r^r yt^r}{\left(c-2xt+yt^2\right)^r}\right]^{-(d+d'-(b+b'))}$$

$$=\left(C-2xt+yt^2\right)^{c-a}\left[1-\frac{r^r yt^r}{\left(C-2xt+yt^2\right)^r}\right]^{-(d-b)}$$

$$\left(C-2xt+yt^2\right)^{c'-a'}\left[1-\frac{r^r yt^r}{\left(C-2xt+yt^2\right)^r}\right]^{-(d'-b')}$$

$$=\sum_{n=0}^{\infty} N_k^{(a,b;c,d)}(x,y,r,C) N_{n-k}^{(a',b';c',d')}(x,y,r,C)t^n.$$

Hence

(4.4.7) $N_n^{(a+a',b+b';c+c'+d+d')}(x,y,r,C)$

$$=\sum_{k=0}^{n} N_k^{(a,b;c,d)}(x,y,r,C) N_{n-k}^{(a',b';c',d')}(x,y,r,C),$$

which can also be generlized in the form

(4.4.8) $N_n^{(a_1+...+a_n,b_1+...+b_n;c_1+...+c_n,d_1+...+d_n)}(x,y,r,C)$

$$=\sum_{i_1+...+i_m=n} n_{i_1}^{(a_1,b_1;c_1,d_1)}(x,y,r,C)...n_{i_m}^{(a_m,b_m;c_m,d_m)}(x,y,r,C).$$

An appeal to generating relation (4.4.2) also shows that

(4.4.9) $N_n^{(a,b;c,d)}(x,y,r,C)=\sum_{k=0}^{n} N_{n-k}^{(a,b;a',b')}(x,y,r,C) N_k^{(a',b';c,d)}(x,y,r,C).$

4.5 The Polynomials $\{R_n^{(a,b;c,d)}(x,y,r,C)/\,n=0,1,2,\ldots\}$. Consider

$$(4.5.1)\; R_n^{(a,b;c,d)}(x,y,r,C)=\sum_{k=0}^{n} A_{n-k}^{(a-k,b)}(x,y,r,C)F_k^{(c,d)}(2,x,y,r,r,C).$$

Therefore,

$$\sum_{n=0}^{\infty} R_n^{(a,b;c,d)}(x,y,r,C)w^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n-k}^{(a-k,b)}(x,y,r,C)F_k^{(c,d)}(2,x,y,r,r,C)w^n$$

$$=\sum_{k=0}^{\infty} F_k^{(c,d)}(2,x,y,r,r,C)w^k \sum_{n=0}^{\infty} A_n^{(a-k,b)}(x,y,r,C)w^n$$

$$=\sum_{k=0}^{\infty} F_K^{(c,d)}(2,x,y,r,r,C)w^k\left(\frac{w}{t}\right)^{a-k}\left(1-r^r yw^r\right)^b$$

$$=\left(\frac{w}{t}\right)^a\left[1-r^r yw^r\right]^b\sum_{k=0}^{\infty}F_k^{(c,d)}(2,x,y,r,C)t^k$$

$$=\left(\frac{w}{t}\right)^a\left[1-r^r yw^r\right]^b\left[C-2xt+yt^2\right]^c\left[1-\frac{r^r yt^r}{\left(C-2xt+yt^2\right)^r}\right]^{-d}$$

$$=\left(\frac{w}{t}\right)^a\left[1-r^r yw^r\right]^b\left(\frac{t}{w}\right)^c\left[1-r^r yw^r\right]^{-d}\quad \text{[Here } \frac{t}{C-2xt+yt^2}=w\,\text{]}$$

$$=\left(\frac{w}{t}\right)^{a-c}\left[1-r^r yw^r\right]^{(-d+b)}$$

$$=\sum_{n=0}^{\infty} A_n^{(a-c,b-d)}(x,y,r,C)w^n.$$

Hence

(4.5.3) $R_n^{(a,b;c,d)}(x,y,r,C)=A_n^{(a-c,b-d)}(x,y,r,C)$.

For $C=a$

(4.5.4) $\sum_{n=0}^{\infty} R_n^{(a,b;a,d)}(x,y,r,C)w^n=[1-r^r y w^r]^{b-d}$

and for $d=b$

(4.5.5) $\sum_{n=0}^{\infty} R_n^{(a,b;c,b)}(x,y,r,C)w^n=\left(\frac{t}{w}\right)^{c-a}$

$$=\left[\frac{2xw+1+\sqrt{(2xw+1)^2-4ycw^2}}{2w^2y}\right]^{c-a}.$$

For $c=a$, $d=b$, (4.5.2.) gives

(4.5.6) $\sum_{n=0}^{\infty} R_n^{(a,b;a,b)}(x,y,r,C)w^n=1$.

Again by an appeal to generating relation (4.5.2), we derive

(4.5.7) $R_n^{(a,b;c,d)}(x,y,r,C)$

$$=\sum_{k=0}^{n} R_{n-k}^{(a,b;e,f)}(x,y,r,C)R_k^{(e,f;c,d)}(x,y,r,C),$$

(4.5.8) $R_n^{(a+a',b+b';c+c',d+d')}(x,y,r,C)$

$$=\sum_{k=0}^{n} R_k^{(a,b;c,d)}(x,y,r,C)R_{n-k}^{(a',b';c',d')}(x,y,r,C),$$

and

(4.5.9) $R_n^{(a_1+...+a_m,b_1+...+b_m;c_1+...+c_m,d_1+...+d_m)}(x,y,r,C)$

$$= \sum_{i_1+...+i_m=n} R_{i_1}^{(a_1,b_1;c_1;d_1)}(x,y,r,C)...R_{i_m}^{(a_m,b_m;c_m,d_m)}(x,y,r,C)$$

4.6. Polynomials $\{A_n^{(p,q)}(x,y,z,r,C)/n=0,1,2,....\}$

Consider

(4.6.1) $\sum_{k=0}^{n} A_k^{(p,q)}(x,y,z,r,C)M_{n-k}^{(p-k,q)}(2,x,y,z,r,r,C)=0$

and

(4.6.2) $A_0^{(p,q)}(x,y,z,r,C)=1$

Therefore,

$$1=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}A_k^{(p,q)}(x,y,z,r,C)M_{n-k}^{(p-k,q)}(2,x,y,z,r,C)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,z,r,C)M_n^{(p-k,q)}(2,x,y,z,r,C)t^{n+k}$$

$$=\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,z,r,C)t^k\sum_{n=0}^{\infty}M_n^{(p-k,q)}(2,x,y,z,r,r,C)t^n$$

$$=\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,z,r,C)t^k\left(C-mxt+yt^2\right)^{p-k}\left[1-\frac{r^rzt^r}{\left(C-2xt+yt^2\right)^r}\right]^{-q}.$$

Hence

(4.6.3) $\left(C-2xt+yt^2\right)^{-p}\left[1-\frac{r^rzt^r}{C-2xt+yt^2}\right]^{q}=\sum_{k=0}^{\infty}A_k^{(p,q)}(x,y,z,r,C)\frac{t^k}{\left(C-2xt+yt^2\right)^k}.$

Therefore, generating relation for $A_n^{(p,q)}$ is

$$(4.6.4)\quad \left(\frac{w}{t}\right)^p \left[1-zr^r w^r\right]^q = \sum_{n=0}^{\infty} A_n^{(p,q)}(x,y,z,r,C)w^n$$

where

$$(4.6.5)\quad \frac{t}{C-2xt+yt^2} = w.$$

The generating relation (4.6.4) can also be written as

$$(4.6.6)\quad \sum_{n=0}^{\infty} A_n^{(p,q)}(x,y,z,r,C)w^n$$

$$= \left[\frac{2xw+1+\sqrt{(2xw+1)^2-4ycw^2}}{2yw^2}\right]^{-p} \left[1-r^r zw^r\right]^q .$$

4.7 Applications of generating relation (4.6.6). An appeal to generating relation (4.6.6) shows that

$$(4.7.1)\quad A_n^{(p+p',q+q')}(x,y,z,r,C) = \sum_{k=0}^{n} A_{n-k}^{(p-q)}(x,y,z,r,C) A_k^{(p',q')}(x,y,z,r,C),$$

$$(4.7.2)\quad A_n^{(p_1+\ldots+p_n,q_1,\ldots,q_m)}(x,y,z,r,C)$$

$$= \sum_{k_1+\ldots+k_m=n} A_{k_1}^{(p,q_1)}(x,y,z,r,C)\ldots A_{k_m}^{(p_m,q_m)}(x,y,z,r,C),$$

$$(4.7.3)\quad A_n^{(p,q-q')}(x,y,z,r,C) = \sum_{k=0}^{[n/r]} A_{n-rk}^{(p,q)} \frac{(q')_q (r^r z)^k}{k!}$$

$$(4.7.4)\quad A_n^{(p,q+1)}(x,y,z,r,C) = A_n^{(p,q)}(x,y,z,r,C) - r^r z A_{n-r}^{(p,q)}(x,y,z,r,C).$$

$$(4.7.5)\quad A_n^{(p,q)}(x,y,z,r,C)=\sum_{k=0}^{n} A_{n-k}^{(p-p',q-q')}(x,y,z,r,C)A_k^{(p',q')}(x,y,z,r,C)$$

$$(4.7.6)\quad A_n^{(p,q)}(x,y,z,r,C)=\sum_{k=0}^{n} A_{n-k}^{(p-p',q')}(x,y,z,r,C)A_k^{(p,q-q')}(x,y,z,r,C)$$

and

$$(4.7.7)\quad A_n^{(p,q)}(x,y,z,r,C)=\sum_{k=0}^{n} A_{n-k}^{(q,p)}(x,y,z,r,C)A_k^{(p-q,q-p)}(x,y,z,r,C).$$

4.8 Differentiation and Applications.

Differentiating (4.6.4) partially with respect to z, we have

$$(4.8.1)\quad \frac{\partial}{\partial z}A_n^{(p,q)}(x,y,z,r,C)=-qr^r A_{n-r}^{(p,q-1)}(x,y,z,r,C).$$

Hence by iteration, we obtain

$$(4.8.2)\quad \frac{\partial^m}{\partial z^m}A_n^{(p,q)}(x,y,z,r,C)=(-1)^m q(q-1)\ldots(q-m+1)A_{n-rm}^{(p,q-m)}(x,y,r,C).$$

Replacing z by $1/z$, we have

$$(4.8.3)\quad \left(z^2\frac{\partial}{\partial z}\right)^m A_n^{(p,q)}(x,y,z,r,C)=q(q-1)\ldots(q-m+1)A_{n-rm}^{(p,q-m)}\left(x,y,\frac{1}{z},r,C\right).$$

Therefore

$$(4.8.4)\quad e^{t\,\Omega_z}\left\{A_n^{(p,q)}(x,y,z,r,C)\right\}=\sum_{m=0}^{\min([n/r],q)} A_{n-rm}^{(p,q-m)}\left(x,y,\frac{1}{z},r,C\right)\binom{q}{m}r^{rm}t^m,$$

where $\Omega_z \equiv z^2\dfrac{\partial}{\partial z}$.

Now making an appeal to

(4.8.5) $\quad e^{t\,\Omega_z}\{f(z)\}=f(z/(1-t))$,

we derive from (4.8.4)

(4.8.6) $$A_n^{(p,q)}(x,y,z-t,r,C)=\sum_{m=0}^{\min([n/r],q)} t^m\binom{q}{m}r^{rm}A_{n-rm}^{(p,q-m)}(x,y,z,r,C).$$

Now taking $z=0$ and replacing t by $(-z)$, we derive

(4.8.7) $$A_n^{(p,q)}(x,y,z,r,C)=\sum_{m=0}^{\min([n/r],q)} z^m\binom{q}{m}r^{rm}A_{n-rm}^{(p,q-m)}(x,y,0,r,C),$$

which can also be obtained by Maclaurine's series expansion.

4.9 The Polynomials $\{N_n^{(a,b;c,d)}(x,y,z,r,C)/\,n=0,1,2,\ldots\}$

Consider

(4.9.1) $$N_n^{(a,b;c,d)}(x,y,z,r,C)=\sum_{k=0}^{n}A_k^{(a,b)}(x,y,z,r,C)M_{n-k}^{(c-k,d)}(2,x,y,z,r,r,C).$$

Therefore

$$\sum_{n=0}^{\infty}N_n^{(a,b;c,d)}(x,y,z,r,C)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_k^{(a,b)}(x,y,z,r,C)M_{n-k}^{(c-k,d)}(2,x,y,z,r,r,C)t^n$$

$$=\sum_{k=0}^{\infty}A_k^{(a,b)}(x,y,z,r,C)t^k\sum_{n=0}^{\infty}M_n^{(c-k,d)}(2,x,y,z,r,r,C)t^n$$

$$=\sum_{k=0}^{\infty}A_k^{(a,b)}(x,y,z,r,C)t^k.(C-2xt+yt^2)^{c-k}\left[1-\frac{r^r z t^r}{(C-2xt+yt^2)^r}\right]^{-d}$$

$$=\left[C-2xt+yt^2\right]^c\left[1-\frac{r^r z t^r}{\left(C-2xt+yt^2\right)^r}\right]^{-d}$$

$$\sum_{k=0}^{\infty} A_k^{(a,b)}(x,y,z,r,C)\left[\frac{t}{C-2xt+yt^2}\right]^k .$$

Now making an appeal to (4.6.3), we derive

(4.9.2) $\sum_{n=0}^{\infty} N_n^{(a,b;c,d)}(x,y,z,r,C)$

$$=\left(C-2xt+yt^2\right)^{c-a}\left[1-\frac{r^r z t^r}{\left(C-2xt+yt^r\right)^r}\right]^{-(d-b)}$$

$$=\sum_{n=0}^{\infty} M_n^{(c-a,d-b)}(2,x,y,z,r,r,C)t^n$$

Hence

(4.9.3) $N_n^{(a,b;c,d)}(x,y,z,r,C)=M_n^{(c-a,d-b)}(2,x,y,z,r,r,C)$.

For $a=c$, (4.9.2) gives

(4.9.4) $\sum_{n=0}^{\infty} N_n^{(a,b;a,d)}(x,y,z,r,C)t^n=\left[1-\frac{r^r z t^r}{\left(C-2xt+yt^2\right)^r}\right]^{b-d}$

For $d=b$, (4.9.2) gives

(4.9.5) $\sum_{n=0}^{\infty} N_n^{(a,b;c,b)}(x,y,z,r,C)t^n=\left(C-2xt+yt^2\right)^{c-a}$.

Again if $a=c$, $b=d$, we derive from (9.2)

(4.9.6) $$\sum_{n=0}^{\infty} N_n^{(a,a;b,b)}(x,y,z,r,C)t^n = 1.$$

Further, making an appeal to generating relation (4.9.2), we establish

(4.9.7) $$N_n^{(a+a',b+b';c+c',d+d')}(x,y,z,r,C)$$

$$= \sum_{k=0}^{n} N_k^{(a,b;c,d)}(x,y,z,r,C).N_{n-k}^{(a',b',c',d')}(x,y,z,r,C),$$

which can be further generalized in the form

(4.9.8) $$N_n^{(a_1+\ldots+a_n,b_1+\ldots+b_n;c_1+\ldots+c_n,d_1+\ldots+d_n)}(x,y,z,r,C)$$

$$= \sum_{i_1+\ldots+i_m=n} N_{i_1}^{(a_1,b_1;c_1,d_1)}(x,y,z,r,C)\ldots N_{i_m}^{(a_m,b_m;c_m,d_m)}(x,y,z,r,C).$$

Again starting wtih generating relation (4.9.2), we can further derive

(4.9.9) $$N_n^{(a,b;c,d)}(x,y,z,r,C) = \sum_{k=0}^{n} N_{n-k}^{(a,b;a',b')}(x,y,z,r,C) N_k^{(a',b';c,d)}(x,y,z,r,C).$$

10. The polynomials $\{R_n^{(a,b;c,d)}(x,y,z,r,C)/n = 0,1,2,3\ldots\}$

Consider

(4.10.1) $$R_n^{(a,b;c,d)}(x,y,z,r,C)$$

$$= \sum_{k=0}^{n} A_{n-k}^{(a-k,b)}(x,y,z,r,C) M_k^{(c,d)}(2,x,y,z,r,r,C).$$

Therefore

$$\sum_{n=0}^{\infty} R_n^{(a,b;c,d)}(x,y,z,r,C)w^n$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_k^{(a-k,b)}(x,y,z,r,C)w^n M_k^{(c,d)}(2,x,y,z,r,r,C)w^k$$

$$=\sum_{k=0}^{\infty} M_k^{(c,d)}(2,x,y,z,r,r,C)w^k.\left(\frac{w}{t}\right)^{a-k}\left(1-zr^r w^r\right)^b \quad \text{[by (4.6.4)]}$$

$$=\left(\frac{w}{t}\right)^a\left(1-zr^r w^r\right)^b\sum_{k=0}^{\infty} M_k^{(c,d)}(2,x,y,z,r,r,C)t^k$$

$$=\left(\frac{w}{t}\right)^a\left[1-zr^r w^r\right]^b\left(C-2xt+yt^2\right)^c\left[1-\frac{r^r zt^r}{\left(C-2xt+yt^2\right)^r}\right]^{-d}$$

$$=\left(\frac{w}{t}\right)^a\left(1-zr^r w^r\right)^b\left(\frac{t}{w}\right)^c\left[1-zr^r w^r\right]^{-d}.$$

Therefore

$$\text{(4.10.2)} \quad \sum_{n=0}^{\infty} R_n^{(a,b;c,d)}(x,y,z,r,C)w^n=\left(\frac{w}{t}\right)^{a-c}\left(1-zr^r w^r\right)^{b-d},$$

which by an appeal to (9.6.4) gives

$$\text{(4.10.3)} \quad R_n^{(a,b;c,d)}(x,y,z,r,C)=A_n^{(a-c,b-d)}(x,y,z,r,C).$$

For $a=c$, (4.10.2) gives

$$\text{(4.10.4)} \quad \sum_{n=0}^{\infty} R_n^{(a,b;a,d)}(x,y,z,r,C)w^n=\left[1-zr^r w^r\right]^{b-d},$$

and for $b=d$, (4.10.2) reduces to

$$\text{(4.10.5)} \quad \sum_{n=0}^{\infty} R_n^{(a,b;c,b)}(x,y,z,r,C)w^n=\left(\frac{t}{w}\right)^{c-a}$$

$$=\left[\frac{2xw+1+\sqrt{(2xw+1)^2-4ycw^2}}{2w^2y}\right]^{c-a},$$

while for c=a, $d=b$, (4.10.2) gives

$$(4.10.6)\quad \sum_{n=0}^{\infty} R_n^{(a,b;c,d)}(x,y,z,r,C)w^n = 1 .$$

Making an appeal to generating relation (4.10.2), we derive

$$(4.10.7)\quad R_n^{(a,b;c,d)}(x,y,z,r,C)$$

$$= \sum_{k=0}^{n} R_{n-k}^{(a,b;e,f)}(x,y,z,r,C) R_k^{(e,f;c,d)} R_k^{(e,f;c,d)}(x,y,z,r,C)$$

and

$$(4.10.8)\quad R_n^{(a+a',b+b';c+c',d+d')}(x,y,z,r,C)$$

$$= \sum_{k=0}^{n} R_{n-k}^{(a,b;c,d)}(x,y,z,r,C) R_k^{(a',b';c',d')}(x,y,z,r,C) .$$

Here (4.10.8) may be further generalized in the form :

$$(4.10.9)\quad R_n^{(a_1+\ldots+a_m,b_1+\ldots+b_m;c_1+\ldots+c_m,d_1+\ldots+d_m)}(x,y,z,r,C)$$

$$= \sum_{i_1+\ldots+i_m=n} R_{i_1}^{(a_1,b_1)}(x,y,z,r,C)..R_{i_m}^{(a_m,b_m)}(x,y,z,r,C) .$$

Remarks

1. Replacing z by z/q and taking $q \to \infty$ our all results of § 6 to § 10 will reduce to the result due to Chandel and Dwivedi [2]
2. For $z=x$, our all results §6 to §10, will reduce to the results due to Chandel and Bhargawa [4].

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GENERATING RELATIONS : TAYLOR'S AND MACLAURINE'S SERIES EXPANSIONS OF HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

CHAPTER -V

GENERATING RELATIONS : TAYLOR'S AND MACLAURINE'S SERIES EXPANSIONS OF HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

5.1 Intoduction. Exton ([5],[6],[7]) introduced twenty one quadruple hypergeometric functions, Sharma and Parihar ([10],[11]) introduced eighty three hypergeometric functions of four variables but out of which nineteen had already been included in the conjecture of Exton ([5],[6],[7]) in some different notations (see Remark of Chandel and Kumar [1]). Also Chandel and Kumar [2] introduced seven more hypergeometric functions of four variables suggested by hypergeometric functions of three variables, H_A, H_B, H_C of Srivastava ([11],[12]).

Further, Chandel and Sharma ([3],[4]) introduced and studied following ten hypergeometric functions of four variables suggested by hypergeometric functions of three variables H_A, H_B, H_C, of Srivastava ([11],[12]).

Further, Chandel and Sharma ([3],[4]) introduced and studied following ten hypergeometric functions of four variables suggested by hypergeometric functions of four variales suggested by hypergeometric functions of three variables H_A, H_B of Srivastava. ([11],[12]). G_A, G_B of Pandey [8] and G_C of Srivastava [13].

(5.1.1) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p}(b)_{m+q}(c)_{p+q}(d)_n}{(e)_{m+p}(e')_n(e'')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

(5.1.2) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+q}(b)_{m+p}(c)_{p+q}(d)_n}{(e_1)_m(e_2)_n(e_3)_p(e_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(5.1.3) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}\frac{(a)_{n+p+q-m}(b)_{m+p}(c)_n(d)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(5.1.4) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}\frac{(a)_{n+p-m}(b)_{m+p+q}(c)_{n+q}(d)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(5.1.5) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}\frac{(a)_{n+p-m}(b)_{m+p}(c)_{n+q}(d)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(5.1.6) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{n+p-m+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(e)_{n+p-m}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

(5.1.7) $G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{n+p-m}(b_1)_{m+q}(b_2)_n(b_3)_p(b_4)_q}{(e)_{n+p-m}(e')_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

(5.1.8) $G_{B_3}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{n+p-m}(b_1)_{n+q}(b_2)_m(b_3)_p(b_4)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}$$

(5.1.9) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+p+q}(b)_{m+n}(c)_{n-p}(d)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}$$

(5.1.10) $G_{C_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+p+q}(b)_{m+p}(c)_{n-p}(d)_q}{(e)_{n+p-m}(e')_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}$$

Recently Chandel and Shamra [4] have established generating relations, integral representations and recurrence relations of above hypergeometric functions of four variables.

In the present Chapter, we shall derive self generating relations cum Taylor's series and Maclaurine's series expansions involving above hypergeometric functions of four variables.

5.2 Generating Relations. Consider

$$\frac{\partial}{\partial x} H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$$

$$= \sum_{m=1}^{\infty} \sum_{n,p,q=0}^{\infty} \frac{(a)_{m+n+p}(b)_{m+q}(c)_{p+q}(d)_n}{(e)_{m+p}(e')_n(e'')_q} \frac{x^{m-1}}{(m-1)!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+1}(b)_{m+q+1}(c)_{p+q}(d)_n}{(e)_{m+p+1}(e')_n(e'')_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}$$

$$= \frac{ab}{e} \sum_{m,n,p,q=0}^{\infty} \frac{(a+1)_{m+n+p}(b+1)_{m+1}(c)_{p+q}(d)_n}{(e+1)_{m+p}(e')_n(e'')_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}.$$

Similarly, by induction, we can derive

(5.2.1) $\dfrac{\partial^r}{\partial x^r} H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$= \frac{(a)_r (b)_r}{(e)_r} \sum_{m,n,p,q=0}^{\infty} \frac{(a+r)_{m+n+p}(b+r)_{m+q}(c)_{p+q}(d)_n}{(e+r)_{m+p}(e')_n(e'')_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

$$= \frac{(a)_r (b)_r}{(e)_r} H_{A_1}^{(4)}(a+r,b+r,c,d;e+r,e',e'';x,y,z,u).$$

Therefore further, by making an appeal to Taylor's series expansion, we establish the following self generating relations for $H_{A_1}^{(4)}$:

(5.2.2) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x+t,y,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} H_{A_1}^{(4)}(a+r,b+r,c,d;e+r,e',e'';x,y,z,u).$$

Applying the same techniques, we derive the following results:

(5.2.3) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y+t,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} H_{A_1}^{(4)}(a+r,b,c,d+r;e,e'+r,e'';x,y,z,u).$$

(5.2.4) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z+t,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} H_{A_1}^{(4)}(a+r,b,c+r,d;e+r,e',e'';x,y,z,u).$$

(5.2.5) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u+t)$

$$= \sum_{r=0}^{\infty} \frac{(b)_r (c)_r}{(e'')_r} \frac{t^r}{r!} H_{A_1}^{(4)}(a,b+r,c+r,d;e,e',e''+r;x,y,z,u).$$

(5.2.6) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x+t,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{t^r}{r!}H_{B_1}^{(4)}(a+r,b+r,c,d;e_1+r,e_2,e_3,e_4;x,y,z,u).$$

(5.2.7) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y+t,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(d)_r}{(e_2)_r}\frac{t^r}{r!}H_{B_1}^{(4)}(a+r,b,c,d+r;e_1,e_2+r,e_3,e_4;x,y,z,u).$$

(5.2.8) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z+t,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e_3)_r}\frac{t^r}{r!}H_{B_1}^{(4)}(a,b+r,c+r,d;e_1,e_2,e_3+r,e_4;x,y,z,u).$$

(5.2.9) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u+t)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e_4)_r}\frac{t^r}{r!}H_{B_1}^{(4)}(a+r,b,c+r,d;e_1,e_2,e_3,e_4+r;x,y,z,u).$$

(5.2.10) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y+t,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e)_r}\frac{t^r}{r!}G_{A_1}^{(4)}(a+r,b,c+r,d;e-r,e';x,y,z,u).$$

(5.2.11) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z+t,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{t^r}{r!}G_{A_1}^{(4)}(a+r,b+r,c,d;e+r,e';x,y,z,u).$$

(5.2.12) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z,u+t)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (d)_r}{(e')_r} \frac{t^r}{r!} G_{A_1}^{(4)}(a+r,b,c,d+r;e,e'+r;x,y,z,u),$$

(5.2.13) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y+t,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (c)_r}{(e)_r} \frac{t^r}{r!} G_{A_2}^{(4)}(a_1+r,b,c+r,d;e+r,e';x,y,z,u),$$

(5.2.14) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z+t,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} G_{A_2}^{(4)}(a+r,b+r,c,d;e+r,e';x,y,z,u),$$

(5.2.15) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z,u+t)$

$$= \sum_{r=0}^{\infty} \frac{(b)_r (c)_r (d)_r}{(e')_r} \frac{t^r}{r!} G_{A_2}^{(4)}(a,b+r,c+r,d+r;e,e'+r;x,y,z,u),$$

(5.2.16) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y+t,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (c)_r}{(e)_r} \frac{t^r}{r!} G_{A_3}^{(4)}(a+r,b,c+r,d;e+r,e';x,y,z,u),$$

(5.2.17) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z+t,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} G_{A_3}^{(4)}(a+r,b+r,c,d;e+r,e';x,y,z,u).$$

(5.2.18) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z,u+t)$

$$= \sum_{r=0}^{\infty} \frac{(c)_r (d)_r}{(e')_r} \frac{t^r}{r!} G_{A_2}^{(4)}(a,b,c+r,d+r;e,e'+r;x,y,z,u),$$

(5.2.19) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y+t,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_2)_r}{(e)_r}\frac{t^r}{r!}G_{B_1}^{(4)}(a+r,b_1,b_2+r,b_3,b_4;e+r,e';x,y,z,u).$$

(5.2.20) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z+t,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_2)_r}{(e)_r}\frac{t^r}{r!}G_{B_1}^{(4)}(a+r,b_1,b_2+r,b_3,b_4;e+r,e';x,y,z,u).$$

(5.2.21) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u+t)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_4)_r}{(e')_r}\frac{t^r}{r!}G_{B_1}^{(4)}(a+r,b_1,b_2,b_3,b_4+r;e,e'+r;x,y,z,u).$$

(5.2.22) $G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y+t,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_2)_r}{(e)_r}\frac{t^r}{r!}G_{B_2}^{(4)}(a+r,b_1,b_2+r,b_3,b_4;e+r,e';x,y,z,u).$$

(5.2.23) $G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z+t,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_3)_r}{(e)_r}\frac{t^r}{r!}G_{B_2}^{(4)}(a+r,b_1,b_2,b_3+r,b_4;e+r,e';x,y,z,u).$$

(5.2.24) $G_{B_3}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u+t)$

$$=\sum_{r=0}^{\infty}\frac{(b_1)_r(b_4)_r}{(e')_r}\frac{t^r}{r!}G_{B_3}^{(4)}(a,b_1+r,b_2,b_3,b_4+r;e,e'+r;x,y,z,u).$$

(5.2.25) $G_{C_1}^{(4)}(a,b,c,d;e,e';x+t,y,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} G_{C_1}^{(4)}(a+r, b+r, c, d; e+r, e'; x, y, z, u),$$

(5.2.26) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y+t,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(b)_r (c)_r}{(e)_r} \frac{t^r}{r!} G_{C_1}^{(4)}(a, b+r, c+r, d; e+r, e'; x, y, z, u),$$

(5.2.27) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u+t)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (d)_r}{(e')_r} \frac{t^r}{r!} G_{C_1}^{(4)}(a+r, b, c, d+r; e, e'+r; x, y, z, u),$$

(5.2.28) $G_{C_2}^{(4)}(a,b,c,d;e,e';x+t,y,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(e)_r} \frac{t^r}{r!} G_{C_2}^{(4)}(a+r, b+r, c, d; e+r, e'; x, y, z, u),$$

(5.2.29) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y+t,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (c)_r}{(e)_r} \frac{t^r}{r!} G_{C_1}^{(4)}(a+r, b, c+r, d; e+r, e'; x, y, z, u).$$

and

(5.2.30) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u+t)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (d)_r}{(e')_r} \frac{t^r}{r!} G_{C_1}^{(4)}(a+r, b, c, d+r; e, e'+r; x, y, z, u).$$

Similarly applying the same techniques we can derive self generating relations-cum- Taylor's series expansions for other hypergeometric functions of four variables due to Exton ([5],[6],[7]) and Sharma-Parihar ([10],[11]). also.

5.3. Maclaurine's Series Expansions. Replacing t by x and x by 0, in (5.2.2), we derive the following Maclaurine's series expansion:

(5.3.1) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e')_r}\frac{x^r}{r!}H(a+r,b+r,c,d;e+r,e',e'';0,y,z,u).$$

Replacing t by y and y by 0 in (5.2.3), we obtain

(5.3.2) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e')_r}\frac{y^r}{r!}H_1^4(a+r,b,c,d+r;e,e'+r,e'';x,0,z,u).$$

Replacing z by zero and t by z in (5.2.4), we get

(5.3.3) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e)_r}\frac{z^r}{r!}H_{A_1}^{(r)}(a+r,b,c+r,d;e+r,e',e'';x,y,0,u).$$

Replacing u by 0 and t by u in (5.2.5), we establish

(5.3.4) $H_{A_1}^{(4)}(a,b,c,d;e,e',e'';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r(c)_r}{(e'')_r}\frac{u^r}{r!}H_{A_1}^{(r)}(a,b+r,c+r,d;e,e',e''+r;x,y,z,0).$$

Replacing x by 0 and t by x in (5.2.6), we arrive at

(5.3.5) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e_1)_r}\frac{x^r}{r!}H_{B_1}^{(4)}(a+r,b+r,c,d;e_1+r,e_2,e_3,e_4;0,y,z,u).$$

Replacing y by 0 and t by y in (5.2.7), we derive

(5.3.6) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(d)_r}{(e_2)_r}\frac{y^r}{r!}H_{B_1}^{(4)}(a+r,b,c,d+r;e_1,e_2+r,e_3,e_4;x,0,z,u).$$

Replacing z by 0 and t by z in (5.2.8), we obtain

(5.3.7) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r(c)_r}{(e_3)_r}\frac{z^r}{r!}H_{B_1}^{(4)}(a,b+r,c+r,d;e_1,e_2,e_3+r,e_4;x,y,0,u).$$

Replcing u by 0 and t by u in (5.2.9), we have

(5.3.8) $H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,e_4;x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e_4)_r}\frac{u^r}{r!}H_{B_1}^{(4)}(a+r,b,c+r,d;e_1,e_2,e_3,e_4+r;x,y,z,0).$$

Replacing y by 0 and t by y in (5.2.10), we establish

(5.3.9) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e)_r}\frac{y^r}{r!}G_{A_1}^{(4)}(a+r,b,c+r,d;e+r,e';x,0,z,u).$$

Replacing z by 0 and t by z in (5.2.10), we arrive at

(5.3.10) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{z^r}{r!}G_{A_1}^{(4)}(a+r,b+r,c,d;e+r,e';x,y,0,u).$$

Replacing u by 0 and t by u in (5.2.11), we obtain

(5.3.11) $G_{A_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(d)_r}{(e')_r}\frac{u^r}{r!}G_{A_1}^{(4)}(a+r,b,c,d+r;e,e'+r;x,y,z,0).$$

Replacing y by 0 and t by y in (5.2.13), we derive

(5.3.12) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e)_r}\frac{y^r}{r!}G_{A_2}^{(4)}(a+r,b,c+r,d;e+r,e';x,0,z,u).$$

Replacing t by z and z by 0 in (5.2.14), we get

(5.3.13) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{z^r}{r!}G_{A_2}^{(4)}(a+r,b+r,c,d;e+r,e';x,y,0,u).$$

Replacing u by 0 and t by u in (5.2.15), we obtain

(5.3.14) $G_{A_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r(c)_r(d)_r}{(e')_r}\frac{u^r}{r!}G_{A_2}^{(4)}(a,b+r,c+r,d+r;e,e'+r;x,y,z,0).$$

Replacing y by 0 and t by y, in (5.2.16), we establish

(5.3.15) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(c)_r}{(e)_r}\frac{y^r}{r!}G_{A_3}^{(4)}(a+r,b,c+r,d;e+r,e';x,0,z,u).$$

Replacing z by 0 and t by z, in (5.2.17), we arrive at

(5.3.16) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{z^r}{r!}G_{A_3}^{(4)}(a+r,b+r,c,d;e+r,e';x,y,0,u).$$

Replacing u *by* 0 and t by u in (5.2.18), we have

(5.3.17) $G_{A_3}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(c)_r(d)_r}{(e')_r}\frac{u^r}{r!}G_{A_3}^{(4)}(a,b,c+r,d+r;e,e'+r;x,y,z,0).$$

Replacing y by *0* and t by y in (5.2.19), we derive

(5.3.18) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_2)_r}{(e)_r}\frac{y^r}{r!}G_{B_1}^{(4)}(a+r,b_1,b_2+r,b_3,b_4;e+r,e';x,0,z,u).$$

Replacing z by *0* and t by z in (5.2.20), we arrive at

(5.3.19) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_2)_r}{(e)_r}\frac{z^r}{r!}G_{B_1}^{(4)}(a+r,b_1,b_2+r,b_3,b_4;e+r,e';x,y,0,u).$$

Replacing t by u and u by *0* in (5.2.21), we get

(5.3.20) $G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_4)_r}{(e')_r}\frac{u^r}{r!}G_{B_1}^{(4)}(a+r,b_1,b_2,b_3,b_4+r;e,e'+r;x,y,z,0).$$

Replacing y by 0 and t by y, in (5.2.22) we have

(5.3.21) $G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_2)_r}{(e)_r}\frac{y^r}{r!}G_{B_2}^{(4)}(a+r,b_1,b_2+r,b_3,b_4;e+r,e';x,0,z,u).$$

Replacing z by 0 and t by z, in (5.2.23), we obtain

(5.3.22) $G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_3)_r}{(e)_r}\frac{z^r}{r!}G_{B_2}^{(4)}(a+r,b_1,b_2,b_3+r,b_4;e+r,e';x,y,0,u).$$

Replacing u by 0 and t by u in (5.2.24), we derive

(5.3.23) $G_{B_3}^{(4)}(a,b_1,b_2,b_3,b_4;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(b_1)_r(b_4)_r}{(e')_r}\frac{u^r}{r!}G_{B_3}^{(4)}(a,b_1+r,b_2,b_3,b_4+r;e,e'+r;x,y,z,0).$$

Replacing x by 0 and t by x in (5.2.25), we establish

(5.3.24) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{x^r}{r!}G_{C_1}^{(4)}(a+r,b+r,c,d;e+r,e';0,y,z,u).$$

Replacing y by 0 and t by y in (5.2.26), we get

(5.3.25) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r(c)_r}{(e)_r}\frac{y^r}{r!}G_{C_1}^{(4)}(a,b+r,c+r,d;e+r,e';x,0,z,u).$$

Replacing u by 0 and t by u in (5.2.27), we arrive at

(5.3.26) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(d)_r}{(e')_r}\frac{u^r}{r!}G_{C_1}^{(4)}(a+r,b,c,d+r;e,e'+r;x,y,z,0).$$

Replacing x by 0 and t by x in (5.2.28), we obtain

(5.3.27) $G_{C_2}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(e)_r}\frac{x^r}{r!}G_{C_2}^{(4)}(a+r,b+r,c,d;e+r,e';0,y,z,u)$$

Replacing y by 0 and t by y in (5.2.29), we have

(5.3.28) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (c)_r}{(e)_r} \frac{y^r}{r!} G_{C_1}^{(4)}(a+r,b,c+r,d;e+r,e';x,0,z,u).$$

Finally, replacing u by 0 and t by u in (5.2.30), we derive

(5.3.29) $G_{C_1}^{(4)}(a,b,c,d;e,e';x,y,z,u)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (d)_r}{(e')_r} \frac{u^r}{r!} G_{C_1}^{(4)}(a+r,b,c,d+r;e,e'+r;x,y,z,0).$$

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GENERATING RELATIONS : TAYLOR'S AND MACLAURINE'S SERIES EXPANSIONS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

CHAPTER -VI

GENERATING RELATIONS : TAYLOR'S AND MACLAURINE'S SERIES EXPANSIONS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

6.1 Intoduction. In the previous Chapter V, we have derived self generating relation cum Taylor's series and Maclaurine's series expansions involving hypergeometric functions of four variables of Chandel and Shamra ([4], [5]).

In the present Chapter, we extend the work by establishing self generating relations cum Taylor's series and Maclaurine's series expansions involving multiple hypergeometric functions of several variables $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ and their confluent forms $\phi_2^{(n)}$, $\psi_2^{(n)}$, $\Xi_1^{(n)}$, $\phi_3^{(n)}$ of Lauricella [8], ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$ of Exton ([6],[7]), ${}^{(k)}_{(1)}E_C^{(n)}$ of Chandel [1], with their confluent forms ${}^{(k)}_{(1)}\phi_{AC}^{(n)}$, ${}^{(k)}_{(2)}\phi_{AC}^{(n)}$, ${}^{(k)}_{(1)}\phi_{AD}^{(n)}$, ${}^{(k)}_{(1)}\phi_{BD}^{(n)}$, ${}^{(k)}_{(2)}\phi_{BD}^{(n)}$ of Chandel and Gupta [2]. ${}^{(k)}F_{CD}^{(n)}$ the intermediate Lauricella's function due to Karlsson [8] and its confluent forms ${}^{(k)}_{(1)}\phi_{CD}^{(n)}$, ${}^{(k)}_{(2)}\phi_{CD}^{(n)}$, ${}^{(k)}_{(3)}\phi_{CD}^{(n)}$, ${}^{(k)}_{(4)}\phi_{CD}^{(n)}$, ${}^{(k)}_{(5)}\phi_{CD}^{(n)}$ due to Chandel and Vishwakarma [3].

6.2. Generating Relations. Consider

$$F_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

Therefore

$$\frac{\partial^r}{\partial x_1^r} F_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_r(b_1)_r}{(c_1)_r} \frac{(a+r)_{m_1+\ldots+m_n}(b_1+r)_{m_1}(b_2)_{m_2}\ldots(b_n)_{m_n}}{(c_1+r)_{m_1}(c_2)_{m_2}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.$$

Hence

$$e^{tD_{x_1}}\left\{F_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b_1)_r}{(c_1)_r} \frac{t^r}{r!} F_A^{(n)}(a+r,b_1+r,b_2,\ldots,b_n;c_1+r,c_2,\ldots,c_n;x_1,\ldots,x_n).$$

Thus we get self-generating function-cum-Taylor series expansion

(6.2.1) $F_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1+t,x_2,\ldots,x_n)$

$$= \sum_{i=0}^{\infty} \frac{(a)_r(b_1)_r}{(c_1)_r} \frac{t^r}{r!} F_A^{(n)}(a+r,b_1+r,b_2,\ldots,b_n;c_1+r,c_2,\ldots,c_n;x_1,\ldots,x_n).$$

which suggests n-results in the following unified form :

(6.2.2) $F_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b_i)_r}{(c_i)_r} \frac{t^r}{r!} F_A^{(n)}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_n)$$

$i=1,\ldots,n.$

Similarly, applying the same techniques, we derive the following generating relations cum Taylor's series expansions of other multiple hypergeometric functions of several variables :

(6.2.3) $F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c,x_1+t,x_2,\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r (b_1)_r}{(c)_r} \frac{t^r}{r!} F_B^{(n)}(a_1 + r, a_2, \ldots, a_n, b_1 + r, b_2, \ldots, b_n; c + r, x_1, \ldots, x_n)$$

which suggests n-results in the following unified form:

(6.2.4) $F_B^{(n)}(a_1, \ldots, a_n, b_1, \ldots, b_n; c, x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a_i)_r (b_i)_r}{(c)_r} \frac{t^r}{r!} F_B^{(n)}(a_1, \ldots, a_{i-1}, a_i + r, a_{i+1}, \ldots, a_n,$$

$$b_1, \ldots, b_{i-1}, b_i + r, b_{i+1}, \ldots, b_n; c + r, x_1, \ldots, x_n),\ |x_1| < 1, \ldots, |x_n| < 1.$$

$i = 1, \ldots, n.$

(6.2.5) $F_C^{(n)}(a, b_i; c_1, \ldots, c_n; x_1 + t, x_2, \ldots, x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c_1)_r} \frac{t^r}{r!} F_C^{(n)}(a + r; b + r; c_1 + r, c_2, \ldots, c_n; x_1, \ldots, x_n),$$

$$|x_1|^{1/2} + \ldots + |x_n|^{1/2} < 1.$$

which suggests n-results of Taylor's series in the following unified form :

(6.2.6) $F_C^{(n)}(a, b; c_1, \ldots, c_n; x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c_i)_r} \frac{t^r}{r!} F_C^{(n)}(a + r; b + r; c_1, \ldots, c_{i-1}, c_i + r, c_{i+1}, \ldots, c_n; x_1, \ldots, x_n)$$

$$|x_1|^{1/2} + \ldots + |x_n|^{1/2} < 1.$$

$i = 1, \ldots, r;$

(6.2.7) $F_D^{(n)}(a; b_1, \ldots, b_n; c; x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b_i)_r}{(c)_r} \frac{t^r}{r!} F_D^{(n)}(a + r; b_1, \ldots, b_{i-1}, b_i + r, b_{i+1}, \ldots, b_n; c + r; x_1, \ldots, x_n)$$

$$|x_1| < 1, \ldots, |x_n| < 1.$$

$i = 1, \ldots, n;$

(6.2.8) $\psi_2^{(n)}(a; c_1,\ldots,c_n; x_1+t, x_2,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_1)_r}\frac{t^r}{r!}\psi_2^{(n)}(a+r; c_1+r, c_2\ldots,c_n; x_1,\ldots,x_n),$$

which suggests n-results of Taylor's series in unified form :

(6.2.9) $\psi_2^{(n)}(a; c_1,\ldots,c_n; x_1,\ldots,x_{i-1}, x_i+t, x_{i+1},\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{t^r}{r!}\psi_2^{(n)}(a+r; c_1,\ldots,c_{i-1}, c_i+r, c_{i+1},\ldots,c_n; x_1,\ldots,x_n),\ i=1,\ldots,n.$$

(6.2.10) $\phi_2^{(n)}(b_1,\ldots,b_n; c; x_1,\ldots,x_{i-1}, x_i+t, x_{i+1},\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{t^r}{r!}\phi_2^{(n)}(b_1,\ldots,b_{i-1}, b_i+r, b_{i+1},\ldots,b_n; c+r; x_1,\ldots,x_n),\ i=1,\ldots,n.$$

(6.2.11) $\Xi_1^{(n)}(a_1,\ldots,a_n, b_1,\ldots,b_{n-1}; c; x_1,\ldots,x_{i-1}, x_i+t, x_{i+1},\ldots,x_n)$

$$=\sum_{i=0}^{\infty}\frac{(a_i)_r}{(b_i)_r(c)_r}\frac{t^r}{r!}\Xi_1^{(n)}(a_1,\ldots,a_{i-1}, a_i+r, a_{i+1},\ldots,a_n; b_1,\ldots,b_{i-1},$$

$$b_i+r, b_{i+1},\ldots,b_{n-1}; c+r; x_1,\ldots,x_n),\quad i=1,\ldots,n-1.$$

(6.2.12) $\phi_3^{(n)}(b_1,\ldots,b_{n-1}; c; x_1,\ldots,x_{i-1}, x_i+t, x_{i+1},\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{t^r}{r!}\phi_3^{(n)}(b_1,\ldots,b_{i-1}, b_i+r, b_{i+1},\ldots,b_{n-1}; c+r; x_1,\ldots,x_n),$$

$$i=1,\ldots,n-1$$

(6.2.13) $\phi_3^{(n)}(b_1,\ldots,b_{n-1}; c; x_1,\ldots,x_{n-1}, x_n+t)$

$$=\sum_{r=0}^{\infty}\frac{1}{(c)_r}\frac{t^r}{r!}\phi_3^{(n)}(b_1,\ldots,b_{n-1}; c+r; x_1,\ldots,x_n),$$

$$(6.2.14)\quad {}^{(k)}_{(1)}E_C^{(n)}(a,a';b;c_1,\dots,c_n;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}E_C^{(n)}(a+r,a';b+r;c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_k,\dots,c_n;$$

$$x_1,\dots,x_i,\dots,x_k,\dots,x_n),\ i=1,\dots,k.$$

$$(6.2.15)\quad {}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\dots,b_n;c,c';x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}E_D^{(n)}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r,c';$$

$$x_1,\dots,x_i,\dots,x_k,\dots,x_n),\ i=1,\dots,k.$$

$$(6.2.16)\quad {}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\dots,b_n;c,c';x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c')_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}E_D^{(n)}(a+r,b_1,\dots,b_k,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c,c'+r;$$

$$x_1,\dots,x_k,\dots,x_i,\dots,x_n),\ i=k+1,\dots,n.$$

$$(6.2.17)\quad {}^{(k)}_{(2)}E_D^{(n)}(a,a';b_1,\dots,b_n;c;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}E_D^{(n)}(a+r,a',b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;$$

$$x_1,\dots,x_i,\dots,x_k,\dots,x_n),\ i=1,\dots,k.$$

$$(6.2.18)\quad {}^{(k)}_{(2)}E_D^{(n)}(a,a',b_1,\dots,b_n;c;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}E_D^{(n)}(a,a'+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;$$

$$x_1,\dots,x_k,\dots,x_i,\dots,x_n),\ i=k+1,\dots,n.$$

(6.2.19) $${}^{(k)}F_{AC}^{(n)}(a,b,b_{k+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}F_{AC}^{(n)}(a+r,b+r,b_{k+1},\dots,b_n;c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;$$

$$x_1,\dots,x_i,\dots,x_k,\dots,x_n),\ i=1,\dots,k.$$

(6.2.20) $${}^{(k)}F_{AC}^{(n)}(a,b,b_{k+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}F_{AC}^{(n)}(a+r,b,b_{k+1},\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c_1,\dots,c_k,\dots,c_{i-1},$$

$$c_i+r,c_{i+1},\dots,c_n;x_1,\dots,x_k,\dots,x_i,\dots,x_n),\ i=k+1,\dots,n.$$

(6.2.21) $${}^{(k)}F_{AD}^{(n)}(a,b_1,\dots,b_n;c,c_{k+1},\dots,c_n;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}F_{AD}^{(n)}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r,c_{k+1},\dots,c_n;x_1,\dots,x_n),$$

$$i=1,\dots,k.$$

(6.2.22) $${}^{(k)}F_{AD}^{(n)}(a,b_1,\dots,b_n;c,c_{k+1},\dots,c_n;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}F_{AD}^{(n)}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;$$

$$c,c_{k+1},\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;x_1,\dots,x_n),\ i=k+1,\dots,n.$$

(6.2.23) $${}^{(k)}F_{BD}^{(n)}(a,a_{k+1},\dots,a_n;b_1,\dots,b_n;c;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}F_{BD}^{(n)}(a+r,a_{k+1},\dots,a_n;b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;x_1,\dots,x_n)$$

$$i=1,\dots,k.$$

(6.2.24) $${}^{(k)}F_{BD}^{(n)}(a,a_{k+1},\dots,a_n;b_1,\dots,b_n;c;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}{}^{(k)}F_{BD}^{(n)}\big(a,a_{k+1},\dots,a_{i-1},a_i+r,a_{i+1},\dots,a_n;b_1,\dots,b_k,\dots,b_{i-1},$$

$$b_i+r,b_{i+1},\dots,b_n;c+r;x_1,\dots,x_n\big),\ i=k+1,\dots,n.$$

(6.2.25) $${}^{(k)}F_{CD}^{(n)}\big(a,;b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}{}^{(k)}F_{CD}^{(n)}\big(a+r;b,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r,c_{k+1},\dots,c_n;x_1,\dots,x_n\big)$$

$$i=1,\dots,k.$$

(6.2.26) $${}^{(k)}F_{CD}^{(n)}\big(a;b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}{}^{(k)}F_{CD}^{(n)}\big(a+r;b+r,b_1,\dots,b_k;c,c_{k+1},\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;$$

$$x_1,\dots,x_n\big)\ i=k+1,\dots,n.$$

(6.2.27) $${}^{(k)}_{(1)}\phi_{AC}^{(n)}\big(a,b;c_1,\dots,c_n;x_1,\dots,x_i+t,\dots,x_k,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}{}^{(k)}_{(1)}\phi_{AC}^{(n)}\big(a+r,b+r;c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_k,\dots,c_n;x_1,\dots,x_n\big)$$

$$i=1,\dots,k.$$

(6.2.28) $${}^{(k)}_{(1)}\phi_{AC}^{(n)}\big(a,b;c_1,\dots,c_n;x_1,\dots,x_k,\dots,x_i+t,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{t^r}{r!}{}^{(k)}_{(1)}\phi_{AC}^{(n)}\big(a+r,b;c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_k,\dots,c_n;x_1,\dots,x_n\big)$$

$$i=k+1,\dots,n.$$

(6.2.29) $${}^{(k)}_{(2)}\phi_{AC}^{(n)}\big(a,b_{k+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{AC}(a+r,b_{k+1},\ldots,b_n;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_k,\ldots,c_n;x_1,\ldots,x_n)$$

$$i=1,\ldots,k.$$

(6.2.30) $${}^{(k)}_{(2)}\phi^{(n)}_{AC}(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{AC}(a+r,b_{k+1},\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c_1,\ldots,c_k,\ldots,c_{i-1},$$

$$c_i+r,c_{i+1},\ldots,c_k,\ldots,c_n;x_1,\ldots,x_n)\quad i=k+1,\ldots,n.$$

(6.2.31) $${}^{(k)}_{(1)}\phi^{(n)}_{AD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_{i-1},x_i+t,x_{i+1}\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{AD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_n)$$

$$i=1,\ldots,k.$$

(6.2.32) $${}^{(k)}_{(1)}\phi^{(n)}_{AD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_{i-1},x_i+t,x_{i+1}\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{AD}(a+r,b_1,\ldots,b_k,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c;x_1,\ldots,x_n)$$

$$i=k+1,\ldots,n.$$

(6.2.33) $${}^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_k,\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{BD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k,\ldots,b_n;c+r;x_1,\ldots,x_n)$$

$$i=1,\ldots,k.$$

(6.2.34) $${}^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_k,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_n)\quad i=k+1,\ldots,n.$$

$$(6.2.35)\quad {}^{(k)}_{(2)}\phi^{(n)}_{BD}\left(a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n\right)$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{BD}\left(a_{k+1},\ldots,a_n,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_n\right)$$

$$i=1,\ldots,k.$$

$$(6.2.36)\quad {}^{(k)}_{(2)}\phi^{(n)}_{BD}\left(a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{BD}\left(a_{k+1},\ldots,a_{i-1},a_i+r,a_{i+1},\ldots,a_n,b_1,\ldots,b_{i-1},b_i+r,\right.$$

$$\left.b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_n\right)\quad i=k+1,\ldots,n.$$

$$(6.2.37)\quad {}^{(k)}_{(1)}\phi^{(n)}_{CD}\left(a,b;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_k,\ldots,x_n\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{CD}\left(a+r,b;c+r,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)\quad i=1,\ldots,k$$

$$(6.2.38)\quad {}^{(k)}_{(1)}\phi^{(n)}_{CD}\left(a,b;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{CD}\left(a+r,b+r;c,c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},c_n;x_1,\ldots,x_n\right)$$

$$i=k+1,\ldots,n$$

$$(6.2.39)\quad {}^{(k)}_{(2)}\phi^{(n)}_{CD}\left(a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}\left(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)$$

$$i=1,\ldots,k.$$

$$(6.2.40)\quad {}^{(k)}_{(2)}\phi^{(n)}_{CD}\left(a,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_n)$$

$i=k+1,\ldots,n.$

(6.2.41) $${}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_k,\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;c+r,c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$$

$i=1,\ldots,k.$

(6.2.42) $${}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\ldots,b_k;c;c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_i,\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(b)_r}{(c_i)_r}\frac{t^r}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{CD}(b+r,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_n)$$

$i=k+1,\ldots,n$

(6.2.43) $${}^{(k)}_{(4)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c;,x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(4)}\phi^{(n)}_{CD}(a+r,b,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;c+r;,x_1,\ldots,x_n)$$

$i=1,\ldots,k.$

(6.2.44) $${}^{(k)}_{(4)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c;,x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}(a)_r(b)_r\frac{t^r}{r!}\,{}^{(k)}_{(4)}\phi^{(n)}_{CD}(a+r,b+r,b_1,\ldots,b_k;c;,x_1,\ldots,x_n)\qquad i=k+1,\ldots,n.$$

(6.2.45) $${}^{(k)}_{(5)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{t^r}{r!}\,{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a+r,b,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$$

$i=1,\ldots,k.$

(6.2.46) $${}^{(k)}_{(5)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a+r,b+r,b_1,\ldots,b_k;c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_n)$$

$$i=k+1,\ldots,n.$$

(6.2.47) $${}^{(k)}_{(6)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_k,\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{t^r}{r!}{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$$

$$i=1,\ldots,k.$$

(6.2.48) $${}^{(k)}_{(6)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{t^r}{r!}{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_k;c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_n)$$

$$i=k+1,\ldots,n.$$

(6.2.49) $${}^{(k)}_{(2)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_k,\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{t^r}{r!}{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k,\ldots,b_n;c_{k+1},\ldots,c_n;$$

$$x_1,\ldots,x_n),\ i=1,\ldots,k.$$

(6.2.50) $${}^{(k)}_{(2)}\phi^{(n)}_{AD}(a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{t^r}{r!}{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_k,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;$$

$$c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_n),\ i=k+1,\ldots,n.$$

(6.2.51) $${}^{(k)}_{(3)}\phi^{(n)}_{CD}(a,a_{k+1},\ldots,a_n,b_1,\ldots,b_k;c;x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_k,\ldots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{AD}(a+r,a_{k+1},\dots,a_n,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r;x_1,\dots,x_n)$$

$i=1,\dots,k.$

(6.2.52) $${}^{(k)}_{(2)}\phi^{(n)}_{BD}(a,a_{k+1},\dots,a_n,b_1,\dots,b_k;c;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{BD}(a,a_{k+1},\dots,a_{i-1},a_i+r,a_{i+1},\dots,a_n,b_1,\dots,b_k;c+r;x_1,\dots,x_n)$$

$i=k+1,\dots,n.$

(6.2.53) $${}^{(k)}_{(1)}\phi^{(n)}_{D}(a,b_1,\dots,b_k;c;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{D}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r;x_1,\dots,x_n)$$

$i=1,\dots,k.$

(6.2.54) $${}^{(k)}_{(1)}\phi^{(n)}_{D}(a,b_1,\dots,b_k;c;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}(a)_r\frac{t^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{D}(a+r,b_1,\dots,b_k;c;x_1,\dots,x_n),\qquad i=k+1,\dots,n$$

(6.2.55) $${}^{(k)}_{(2)}\phi^{(n)}_{D}(a,b_1,\dots,b_n;c;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{D}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k,\dots,b_n;c+r;x_1,\dots,x_n).$$

$i=1,\dots,k.$

(6.2.56) $${}^{(k)}_{(2)}\phi^{(n)}_{D}(a,b_1,\dots,b_n;c;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{t^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{D}(a,b_1,\dots,b_k,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;x_1,\dots,x_n).$$

$$i=k+1,\dots,n.$$

(6.2.57) $_{(1)}^{(k)}\phi_C^{(n)}(a,b;c_1,\dots,c_n;x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_k,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{t^r}{r!}\,{}_{(1)}^{(k)}\phi_C^{(n)}(a+r,b+r;c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;x_1,\dots,x_n)$$

$$i=1,\dots,k.$$

(6.2.58) $_{(1)}^{(k)}\phi_C^{(n)}(a,b;c_1,\dots,c_n;x_1,\dots,x_k,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r}{(c_i)_r}\frac{t^r}{r!}\,{}_{(1)}^{(k)}\phi_C^{(n)}(a,b+r;c_1,\dots,c_k,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;x_1,\dots,x_n),$$

$$i=k+1,\dots,n.$$

6.3 Maclaurine's Series Expansions. In this Secion, we shall derive Maclaurine's series expansions for multiple hypergeometric functions of several variables from the results established in previous Section 6.2.

Replacing t by x_1 and x_1 by zero in (6.2.1), we derive the Maclaurine's series

(6.3.1) $F_A^{(n)}(a,b_1,\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_1)_r}{(c_1)_r}\frac{x_1^r}{r!}F_A^{(n)}(a+r,b_1+r,b_2,\dots,b_n;c_1+r,c_2,\dots,c_n;0,x_2,\dots,x_n)$$

The result (6.3.1) suggests n-results in the following unified form:

(6.3.2) $F_A^{(n)}(a,b_1,\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{x_i^r}{r!}F_A^{(n)}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;$$

$$c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),|x_1|+\dots+|x_n|<1,\ i=1,\dots,n.$$

Replacing t by x_i and x_i by zero in (6.2.4), we derive

(6.3.3) $F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}F_B^{(n)}(a_1,\ldots,a_{i-1},a_i+r,a_{i+1},\ldots,a_n;b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;$$

$$c+r;x_1,\ldots,x_{i-1},0,x_{i+1},x_n),\qquad |x_1|<1\ ,\ldots,|x_n|<1;\ i=1,\ldots n.$$

Replacing t by x_i and x_i by zero in (6.2.6), we derive

(6.3.4) $F_C^{(n)}(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^r}{r!}F_C^{(n)}(a+r;b+r;c_1,\ldots,c_{i-1},c_i+r,c_{i+1}\ldots,c_n;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)$$

$$|x_1|^{1/2}+\ldots+|x_n|^{1/2}<1;\ i=1,\ldots n.$$

Replacing t by x_i and x_i by zero in (6.2.7), we obtain

(6.3.5) $F_D^{(n)}(a;b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}F_D^{(n)}(a+r;b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)$$

$$|x_1|<1\ \ldots|x_n|<1;\ i=1,\ldots n.$$

Replacing t by x_i and x_i by zero in (6.2.9), we get

(6.3.6) $\psi_2^{(n)}(a;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{i=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{x_i^r}{r!}\psi_2^{(n)}(a+r;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_{i-1},0,x_{i+1},x_n),$$

$$i=1,\ldots,n.$$

Replacing t by x_i and x_i by zero in (6.2.10), we establish

(6.3.7) $\phi_2^{(n)}(b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r}{(c)_r} \frac{x_i^r}{r!} \phi_2^{(n)}(b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),$$

$i=1,\ldots,n.$

Replacing t by x_i and x_i by zero in (6.2.11), we obtain

(6.3.8) $\Xi_1^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_{n-1};c;x_1,\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a_i)_r}{(b_i)_r(c)_r} \frac{t^r}{r!} \Xi_1^{(n)}(a_1,\ldots,a_{i-1},a_i+r,a_{i+1},\ldots,a_n;b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_{n-1};$$

$$c+r;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\qquad i=1,\ldots,n.$$

Replacing t by x_i and x_i by zero in (6.2.12), we get

(6.3.9) $\phi_3^{(n)}(b_1,\ldots,b_{n-1};c;x_1,\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r}{(c)_r} \frac{x_i^r}{r!} \phi_3^{(n)}(b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_{n-1};c+r;x_1,\ldots,x_{i-1},0,x_{i+1},x_n)$$

$i=1,\ldots,n-1$

Also replacing t by x_i and x_i by zero in (6.2.13), we arrive at

(6.3.10) $\phi_3^{(n)}(b_1,\ldots,b_{n-1};c;x_1,\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{1}{(c)_r} \frac{x_i^r}{r!} \phi_3^{(n)}(b_1,\ldots,b_{n-1};c+r;x_1,\ldots,x_{n-1},0).$$

Similarly replacing t by x_i and x_i by zero in (6.2.14) to (6.2.58), we derive the following results respectively :

(6.3.11) ${}^{(k)}_{(1)}E_C^{(n)}(a,a';b;c_1,\ldots,c_n;x_1,\ldots,x_k,\ldots,x_i,\ldots,x_n)$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c_i)_r} \frac{x_i^r}{r!} {}^{(k)}_{(1)}E_C^{(n)}(a+r,a',b+r;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_k,\ldots,c_n;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k,\dots,x_n) \quad i=1,\dots,k.$$

(6.3.12) $${}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\dots,b_n;c,c';x_1,\dots,x_i,\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}E_D^{(n)}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r,c';$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k,\dots,x_n), \qquad i=1,\dots,k.$$

(6.2.13) $${}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\dots,b_n;c,c';x_1,\dots,x_k,\dots,x_i,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c')_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}E_D^{(n)}(a+r,b_1,\dots,b_k,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c,c'+r;$$

$$x_1,\dots,x_k,\dots,x_{i-1},0,x_{i+1},\dots,x_n) \quad i=k+1,\dots,n$$

(6.3.14) $${}^{(k)}_{(2)}E_D^{(n)}(a,a';b_1,\dots,b_n;c;x_1,\dots,x_i,\dots,x_k,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}E_D^{(n)}(a+r,a',b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k,\dots,x_n), \quad i=1,\dots,k.$$

(6.3.15) $${}^{(k)}_{(2)}E_D^{(n)}(a,a';b_1,\dots,b_n;c;x_1,\dots,x_k,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}E_D^{(n)}(a,a'+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;$$

$$x_1,\dots,x_k,\dots,x_{i-1},0,x_{i+1},\dots,x_n) \quad i=k+1,\dots,n.$$

(6.3.16) $${}^{(k)}F_{AC}^{(n)}(a,b,b_{k+1},\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}F_{AC}^{(n)}(a+r,b+r,b_{k+1},\dots,b_n;c_1,\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k,\ldots,x_n)\quad i=1,\ldots,k.$$

(6.3.17) ${}^{(k)}F_{AC}^{(n)}(a,b,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{x_i^r}{r!}\,{}^{(k)}F_{AC}^{(n)}(a+r,b,b_{k+1},\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c_1,\ldots,c_k,\ldots,c_{i-1},$$

$$c_i+r,c_{i+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\ i=k+1,\ldots,n.$$

(6.3.18) ${}^{(k)}F_{AD}^{(n)}(a,b_1,\ldots,b_n;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}\,{}^{(k)}F_{AD}^{(n)}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r,c_{k+1},\ldots,c_n$$

$$;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k,\ldots,x_n),\ i=1,\ldots,k.$$

(6.3.19) ${}^{(k)}F_{AD}^{(n)}(a,b_1,\ldots,b_n;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_k,\ldots,x_i,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{x_i^r}{r!}\,{}^{(k)}F_{AD}^{(n)}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c,c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n$$

$$;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\quad i=k+1,\ldots,n.$$

(6.3.20) ${}^{(k)}F_{BD}^{(n)}(a,a_{k+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}\,{}^{(k)}F_{BD}^{(n)}(a+r,a_{k+1},\ldots,a_n,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k,\ldots,b_n;$$

$$c+r;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\ i=1,\ldots,k.$$

(6.3.21) ${}^{(k)}F_{BD}^{(n)}(a,a_{k+1},\ldots,a_n;b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}\,{}^{(k)}F_{BD}^{(n)}(a,a_{k+1},\ldots,a_{i-1},a_i+r,a_{i+1},\ldots,a_n;b_1,\ldots,b_k,\ldots,b_{i-1},$$

$$b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n\Big),\quad i=k+1,\ldots,n.$$

(6.3.22) ${}^{(k)}F_{CD}^{(n)}\left(a;b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}F_{CD}^{(n)}\Big(a+r;b,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;$$

$$c+r,c_{k+1},\ldots,c_n;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k,\ldots,x_n\Big),\quad i=1,\ldots,k.$$

(6.3.23) ${}^{(k)}F_{CD}^{(n)}\left(a,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;x_1,\ldots,x_n\right)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}F_{CD}^{(n)}\Big(a+r;b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;$$

$$x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n\Big),\quad i=k+1,\ldots,n.$$

(6.3.24) ${}^{(k)}_{(1)}\phi_{4C}^{(n)}\left(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi_{4C}^{(n)}\Big(a+r,b+r;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_k,\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n\Big),\quad i=1,\ldots,k.$$

(6.3.25) ${}^{(k)}_{(1)}\phi_{4C}^{(n)}\left(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi_{4C}^{(n)}\Big(a+r,b;c_1,\ldots,c_k,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n\Big),\quad i=k+1,\ldots,n.$$

(6.3.26) ${}^{(k)}_{(2)}\phi_{4C}^{(n)}\left(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi_{4C}^{(n)}\Big(a+r,b_{k+1},\ldots,b_n;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_k,\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\quad i=1,\ldots,k.$$

(6.3.27) $^{(k)}_{(2)}\phi^{(n)}_{AC}(a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{AC}(a+r,b_{k+1},\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c_1,\ldots,c_k,\ldots,c_{i-1},$$

$$c_i+r,c_{i+1},\ldots,c_n;\,x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\quad i=k+1,\ldots,n.$$

(6.3.28) $^{(k)}_{(1)}\phi^{(n)}_{AD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{AD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k,\ldots,x_n)\,i=1,\ldots,k.$$

(6.3.29) $^{(k)}_{(1)}\phi^{(n)}_{AD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{AD}(a+r,b_1,\ldots,b_k,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c;$$

$$x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\,i=k+1,\ldots,n.$$

(6.3.30) $^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{BD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k,\ldots,b_n;c+r;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\quad i=1,\ldots,k.$$

(6.3.31) $^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_k,\ldots,x_i,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{BD}(a,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)$$

$$i=k+1,\dots,n.$$

(6.3.32) $${}^{(k)}_{(2)}\phi^{(n)}_{BD}(a_{k+1},\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{BD}(a_{k+1},\dots,a_n,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=1,\dots,k.$$

(6.3.33) $${}^{(k)}_{(2)}\phi^{(n)}_{BD}(a_{k+1},\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{BD}(a_{k+1},\dots,a_{i-1},a_i+r,a_{i+1},\dots,a_n,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;$$

$$c+r;x_1,\dots,x_k,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=k+1,\dots,n.$$

(6.3.34) $${}^{(k)}_{(1)}\phi^{(n)}_{CD}(a,b;c,c_{k+1},\dots,c_n;x_1,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{CD}(a+r,b;c+r,c_{k+1},\dots,c_n;x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=1,\dots,k$$

(6.3.35) $${}^{(k)}_{(1)}\phi^{(n)}_{CD}(a,b;c,c_{k+1},\dots,c_n;x_1,\dots,x_k,\dots,x_i,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{CD}(a+r,b+r;c,c_{k+1},\dots,c_{i-1},c_i+r,c_{i+1},c_n;$$

$$x_1,\dots,x_k,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=k+1,\dots,n.$$

(6.3.36) $${}^{(k)}_{(2)}\phi^{(n)}_{CD}(a,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;c+r,c_{k+1},\dots,c_n;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=1,\dots,k.$$

(6.3.37) $ {}^{(k)}_{(2)}\phi^{(n)}_{CD}(a,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a+r,b_1,\dots,b_k;c,c_{k+1},\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=k+1,\dots,n.$$

(6.3.38) $ {}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r,c_{k+1},\dots,c_n;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=1,\dots,k.$$

(6.3.39) $ {}^{(k)}_{(3)}\phi^{(n)}_{CD}(b,b_1,\dots,b_k;c,c_{k+1},\dots,c_n;x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{CD}(b+r,b_1,\dots,b_k;c,c_{k+1},\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\quad i=k+1,\dots,n$$

(6.3.40) $ {}^{(k)}_{(4)}\phi^{(n)}_{CD}(a,b,b_1,\dots,b_k;c;x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(4)}\phi^{(n)}_{CD}(a+r,b,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),\ i=1,\dots,k$$

(6.3.41) $ {}^{(k)}_{(4)}\phi^{(n)}_{CD}(a,b,b_1,\dots,b_k;c;,x_1,\dots,x_n)$

$$=\sum_{r=0}^{\infty}(a)_r(b)_r\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(4)}\phi^{(n)}_{CD}(a+r,b+r,b_1,\dots,b_k;c;,x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n),$$

$$i=k+1,\dots,n$$

(6.3.42) $\,{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a+r,b,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;c_{k+1},\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\; i=1,\ldots,k$$

(6.3.43) $\,{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(5)}\phi^{(n)}_{CD}(a+r,b+r,b_1,\ldots,b_k;c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\; i=k+1,\ldots,n.$$

(6.3.44) $\,{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k;c_{k+1},\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k,\ldots,x_n),\; i=1,\ldots,k.$$

(6.3.45) $\,{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_k;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(6)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_k;c_{k+1},\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;$$

$$x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\; i=k+1,\ldots,n.$$

(6.3.46) $\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a,b_1,\ldots,b_n;c_{k+1},\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}(a)_r(b_i)_r\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{CD}(a+r,b_1,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_k,\ldots,b_n;c_{k+1},\ldots,c_n;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n\big),\quad i=1,\dots,k$$

(6.3.47) $${}^{(k)}_{(2)}\phi^{(n)}_{AD}\big(a,b_1,\dots,b_n;c_{k+1},\dots,c_n;x_1,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c_i)_r}\frac{x_i^r}{r!}\,{}^{(k)}_{(2)}\phi^{(n)}_{AD}\big(a+r,b_1,\dots,b_k,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_n;$$

$$c_{k+1},\dots,c_{i-1},c_i+r,c_{i+1},\dots,c_n;x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n\big),\quad i=k+1,\dots,n$$

(6.3.48) $${}^{(k)}_{(3)}\phi^{(n)}_{CD}\big(a,a_{k+1},\dots,a_n,b_1,\dots,b_k;c;x_1,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{AD}\big(a+r,a_{k+1},\dots,a_n,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k,\dots,x_n\big),\quad i=1,\dots,k.$$

(6.3.49) $${}^{(k)}_{(3)}\phi^{(n)}_{BD}\big(a,a_{k+1},\dots,a_n,b_1,\dots,b_k;c;x_1,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a_i)_r}{(c)_r}\frac{x_i^r}{r!}\,{}^{(k)}_{(3)}\phi^{(n)}_{BD}\big(a,a_{k+1},\dots,a_{i-1},a_i+r,a_{i+1},\dots,a_n,b_1,\dots,b_k;c+r;$$

$$x_1,\dots,x_k,\dots,x_{i-1},0,x_{i+1},\dots,x_n\big),\quad i=k+1,\dots,n$$

(6.3.50) $${}^{(k)}_{(1)}\phi^{(n)}_{D}\big(a,b_1,\dots,b_k;c;x_1,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{D}\big(a+r,b_1,\dots,b_{i-1},b_i+r,b_{i+1},\dots,b_k;c+r;$$

$$x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k,\dots,x_n\big),\quad i=1,\dots,k$$

(6.3.51) $${}^{(k)}_{(1)}\phi^{(n)}_{D}\big(a,b_1,\dots,b_k;c;x_1,\dots,x_n\big)$$

$$=\sum_{r=0}^{\infty}(a)_r\frac{x_i^r}{r!}\,{}^{(k)}_{(1)}\phi^{(n)}_{D}\big(a+r,b_1,\dots,b_k;c;x_1,\dots,x_k,\dots,x_{i-1},0,x_{i+1},\dots,x_n\big)$$

$$i=k+1,\ldots,n$$

(6.3.52) ${}^{(k)}_{(2)}\phi_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi_D^{(n)}(a+r,b_1,\ldots,b_k,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;x_1,\ldots,x_n)$$

$$i=k+1,\ldots,n.$$

(6.3.53) ${}^{(k)}_{(2)}\phi_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b_i)_r}{(c)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(2)}\phi_D^{(n)}(a,b_1,\ldots,b_k,\ldots,b_{i-1},b_i+r,b_{i+1},\ldots,b_n;c+r;$$

$$x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\ i=k+1,\ldots,n.$$

(6.3.54) ${}^{(k)}_{(1)}\phi_C^{(n)}(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi_C^{(n)}(a+r,b+r;c_1,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;$$

$$x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k,\ldots,x_n),\ i=1,\ldots,k$$

(6.3.55) ${}^{(k)}_{(1)}\phi_C^{(n)}(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n)$

$$=\sum_{r=0}^{\infty}\frac{(b)_r}{(c_i)_r}\frac{x_i^{\,r}}{r!}\,{}^{(k)}_{(1)}\phi_C^{(n)}(a,b+r;c_1,\ldots,c_k,\ldots,c_{i-1},c_i+r,c_{i+1},\ldots,c_n;$$

$$x_1,\ldots,x_k,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),\ i=k+1,\ldots,n.$$

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A GENERAL CLASS OF GENERATING FUNCTIONS THROUGH GROUP THEORETIC APPROACH AND ITS APPLICATIONS

CHAPTER -VII

A GENERAL CLASS OF GENERATING FUNCTIONS THROUGH GROUP THEORETIC APPROACH AND ITS APPLICATIONS

In the present Chapter, we introduce a general class of generating functions involving the product of modified Bessel polynonomials $Y_n^{\alpha+n}(.)$ and the confluent hypergeometric function ${}_1F_1$ (.) and then, obtain its some more general class of generating functions by group-theoretic approach and discuss their applications.

7.1. Introduction. In 1949, Krall and Frink [3] introduced generalized Bessel polynomials defined by

(7.1.1) $Y_n^{\alpha}(x) = {}_2F_0[-n, n+\alpha-1; -; -x/\beta]$

Further, in 1987, Mukherjee and Chongdar [5] have considered and studied the modified Bessel polynomials defined by

(7.1.2) $Y_n^{\alpha+n}(x) = {}_2F_0[-n, 2n+\alpha-1; -; -x/\beta]$

The function ${}_1F_1(.)$ can be replaced by many special functions such as the Laguerre polynomials or the parabolic cylinder functions etcetera.

Srivastava and Manocha [6] defined and studied various bilinear, bilateral and multilinear generating functions. Further, in 1989, Chatterjee and Chakraborty [1] introduced and studied some quasi-bilinear and quasi-bilateral generating functions.

Motivated by above work, in the present Chapter, we introduce the following new general class of generating functions :

(7.1.3) $G(x,u,w) = \sum_{n=0}^{\infty} A_n w^n \, Y_n^{\alpha+n}(x) \; {}_1F_1\left[\begin{matrix} -n; \\ n+1; \end{matrix} u\right]$

where, A_n is any arbitrary sequence independent of x, u and w. Since in (7.13) as setting various values of An, we may find several results on generating functions involving different special functions.

Further, making an appeal to the group-theoretic techniques, here in the present Chapter, we evaluate some more general class of generating functions and finally discuss their applications.

7.2. Group-Theoretic Operetors. In our investigations, we use the following group- theoretic-operators and their actions :

The operator H_1 due to Kar [2] is given by

$$(7.2.1)\ H_1 = x^2yz^{-2}\frac{\partial}{\partial x} + 2xy^2z^{-2}\frac{\partial}{\partial y} + xyz^{-1}\frac{\partial}{\partial z} + (\beta - x)yz^{-2}$$

such that

$$(7.2.2)\ H_1\left[Y_n^{\alpha+n}(x)y^nz^{\alpha}\right] = \beta y_{n+1}^{\alpha+n-1}(x)y^{n+1}z^{\alpha'-2}.$$

The operator H_2 due to Miller Jr. [4] is given by

$$(7.2.3)\ H_2 = v\frac{\partial}{\partial t} + vut^{-1}\frac{\partial}{\partial u} - vut^{-1}$$

such that

$$(7.2.4)\ H_2\left[{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}\ u\right]v^nt^m\right] = m\,{}_1F_1\left[\begin{matrix}-n-1;\\ m;\end{matrix}\ u\right]v^{n+1}t^{m-1}.$$

The actions of H_1 and H_2 on f are obtained as follows :

$$(7.2.5)\exp[wH_1]f(x,y,z) = \left(1-\frac{wxy}{z^2}\right)\exp\left[\frac{w\beta y}{z^2}\right]f\left(\frac{x}{1-\frac{wxy}{z^2}}, \frac{y}{\left(1-\frac{wxy}{z^2}\right)^2}, \frac{z}{1-\frac{wxy}{z^2}}\right)$$

and

$$(7.2.6)\ \exp[wH_2]f(v,t,u) = \exp\left[\frac{-uvw}{t}\right]f\left(v, t+wv, u\left(1+\frac{wv}{t}\right)\right)$$

7.3 Some more general class of generating furnction

In this section, making an use of the general class of generating function (7.1.3) and group-theoratic operators H_1 and H_2 with their actions given in the Section 7.2, we obtain some more general class of generating functions through following theorem :

Theorem. *If there exists a general class of generating functions involving the product of modified Bessel polynomials and the confluent hypergeometric functions given by* (7.1.3), *then following more general class of generating functions hold :*

$$(7.3.1)\quad (1+w)^m (1-wx)^{1-\alpha} \exp[w(\beta-u)] G\left[\frac{x}{1-wx}, u(1+w), \frac{wyt}{(1-wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_n \frac{(-1)^s \Gamma(m+1)}{r!s!\Gamma(m-s+1)} Y_{n+r}^{\alpha+n-r}(x)\, {}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}\, u\right](wyt)^n (\beta w)^r (-w)^s$$

$$(7.3.2)\quad (1+w)^m (1+wx)^{1-\alpha} \exp[-w(\beta+u)] G\left[\frac{x}{1+wx}, u(1+w), \frac{wyt}{(1+wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_n \frac{(-1)^s \Gamma(m+1)}{r!s!\Gamma(m-s+1)} Y_{n+r}^{\alpha+n-r}(x)\, {}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}\, u\right](wyt)^n (-\beta w)^r (-w)^s .$$

or equivalently,

$$(7.3.3)\quad (1+w)^m (1-wx)^{1-\alpha} \exp[w(\beta-u)] G\left[\frac{x}{1-wx}, u(1+w), \frac{wyt}{(1-wx)^2}\right],$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{r} A_n \frac{(-r)_n (-1)^s \Gamma(m+1)}{r!s!\Gamma(m-s+1)} Y_{n+r}^{\alpha+2n-r}(x)\, {}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}\, u\right](-yt)^n (\beta)^{r-n} (-w)^s w^r$$

and

(7.3.4) $(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{r}A_n\frac{(-r)_n(-m)_s}{r!s!}Y_r^{\alpha+2n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](yt)^n(\beta)^{r-n}(-w)^s(-w^r).$$

Proof : In the general class of generating function (7.1.3), replacing w by wyv and then multiplying by $z^\alpha t^m$ both sides, we get

(7.3.5) $G(x,u,wyv)z^\alpha t^m=\sum_{n=0}^{\infty}A_nw^ny_n^{\alpha+n}(x)y^nz^\alpha\,{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}u\right]v^nt^m.$

Now, making an appeal to (7.2.2) and (7.2.4), from (7.3.5), we derive

(7.3.6) $\exp[wH_1]\{Y_n^{\alpha+n}(x)y^nz^\alpha\}=\sum_{n=0}^{\infty}\frac{(w)^r}{r!}\beta^ry_{n+r}^{\alpha+n-r}(x)y^{n+r}z^{\alpha-2r}$

and

(7.3.7) $\exp[wH_2]\left\{{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}u\right]v^nt^m\right\}$

$$=\sum_{s=0}^{\infty}\frac{(-w)^s}{s!}\frac{(-1)^s\Gamma(m+1)}{\Gamma(m-s+1)}{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]v^{n+s}t^{m-s}.$$

Now, operating both sides of (7.3.5) by the operators $exp[wH_1]exp[wH_2]$ and then, making an appeal to the relations (7.2.5) and (7.2.6) in the left hand side and (7.3.5) and (7.3.6) in the right hand side, we evaluate

(7.3.8) $z^\alpha(t+wv)^m(1-wxy/z^2)^{1-\alpha}\exp[w(\beta y/z^2-vu/t)]$

$$G\left[\frac{x}{1-wxy/z^2},u(1+wv/t),\frac{wyv}{(1-wxy/z^2)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}A_n\frac{(w)^{n+r+s}}{r!s!}\frac{\Gamma(m+1)}{\Gamma(m-s+1)}Y_{n+r}^{\alpha+n-r}(x)y^{n+r}z^{\alpha-2r}{}_1F_1\left[\begin{matrix}-n-s;\\m-s+1;\end{matrix}u\right]v^{n+s}t^{m-s}.$$

Now, setting $y/z^2=1$ and $v=t$ in (7.3.8), we prove (7.3.1).

Again, setting $y/z^2=-1$ and $v=t$ in (7.3.8), we prove (7.3.2).

Finally, replacing r by $r-n$ and than applying series rearrangement techniques in (7.3.1) and (7.3.2), we obtain (7.3.3) and (7.3.4) reepectively.

7.4 Special Cases : Applications and Deductions.

For m a positive integer, (7.3.1), (7.3.2), (7.3.3) and (7.3.4) reduce respectively to

(7.4.1) $$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{(-m)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\m-s+1;\end{matrix}u\right](wyt)^n(\beta w)^r(-w)^s$$

(7.4.2) $$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{(-m)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\m-s+1;\end{matrix}u\right](wyt)^n(-\beta w)^r(-w)^s,$$

(7.4.3) $$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{s=0}^{m}A_n\frac{(-r)_n(-m)_s}{r!s!}Y_r^{\alpha+m-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\m-s+1;\end{matrix}u\right](-yt)^n(\beta)^{r-n}(w)^r$$

and

(7.4.4) $(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{s=0}^{m}A_n\frac{(-r)_n(-m)_s}{r!s!}Y_r^{\alpha+2n-s}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}\,u\right](yt)^n(\beta)^{r-n}(-w)(-w)^r.$$

From (7.4.1), we further derive

(7.4.5) $(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{A_{n-r}(-m)_s}{r!s!}Y^{\alpha+n-2r}(x)\,{}_1F_1\left[\begin{matrix}-(n+s-r);\\ m-s+1;\end{matrix}\,u\right](yt)^{n-r}w^n(-w)^s(\beta w)^r,$$

while from (7.4.2), we obtain

(7.4.6) $(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{A_{n-r}(-m)_s}{r!s!}Y_n^{\alpha+n-2r}(x)\,{}_1F_1\left[\begin{matrix}-(n-r+s);\\ m-s+1;\end{matrix}\,u\right](yt)^{n-r}w^n(-\beta)^r(-w)^s.$$

Further setting $\beta=u$ and $t=1$ in (7.3.1), we derive

(7.4.7) $(1+w)^m(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{m!n!}{(m+n)!}\frac{(1+n)_s}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s,$$

where $L_n^{(m)}(u)$ are Laguerre polynomials.

For $\beta=-u$, $t=1$, (7.3.2) gives

(7.4.8) $$(1+w)^{m}(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^{2}}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_{n}\frac{m!n!}{(m+n)!}\frac{(1+n)_{s}}{r!s!}y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^{n}(uw)^{r}(w)^{s}.$$

Other similar results can be obtained form (7.3.3) and (7.3.4) in similar manner.

If m is positive integer than (7.4.7) and (7.4.8) give

(7.4.9) $$(1+w)^{m}(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^{2}}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_{n}\frac{m!n!}{(m+n)!}\frac{(1+n)_{s}}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^{n}(uw)^{r}(w)^{s}$$

and

(7.4.10) $$(1+w)^{m}(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^{2}}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_{n}\frac{m!n!}{(m+n)!}\frac{(1+n)_{s}}{r!s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^{n}(uw)^{r}(w)^{s}$$

respectively.

Further setting $m=0$ and $t=1$ in (7.3.1), we derive a generating relation

(7.4.11) $$(1-wx)^{1-\alpha}\exp(\beta w)G\left[\frac{x}{1-wx},\frac{wy}{(1-wx)^{2}}\right]$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}A_{n}\frac{w^{r}}{(r-n)!}\beta^{r-n}Y_{r}^{\alpha+2n-r}(x)y^{n},$$

which is similar result due to Mukherjee and Chongdar [5],

while for $m=0$ and $t=1$, from (7.3.2), we obtain a generating relation :

$$(7.4.12)\quad (1+wx)^{1-\alpha}\exp(-\beta w)G\left[\frac{x}{1+wx},\frac{wy}{(1+wx)^2}\right]$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}A_n\frac{w^r}{(r-n)!}(-\beta)^{r-n}y_r^{\alpha+2n-r}(x)y^n .$$

From (7.4.7) and (7.4.8), we further derive a relation

$$(7.4.13)\quad (1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$$

$$=(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right].$$

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APPLICATIONS OF GENERALIZED POLYNOMIALS OF SEVERAL VARIABLES AND MULTIVARIABLE *H*-FUNCTION OF SRIVASTAVA-PANDA IN TWO BOUNDARY VALUE PROBLEMS

APPLICATIONS OF GENERALIZED POLYNOMIALS OF SEVERAL VARIABLES AND MULTIVARIABLE *H*-FUNCTION OF SRIVASTAVA-PANDA IN TWO BOUNDARY VALUE PROBLEMS

In the present Chapter, first we evaluate an integral involving product of the general class of polynomials of Srivastava [6] and multivariable *H*-function of Srivastava and Panda ([10], [11] and also see [12]) and then make its applications to solve two boundary value problems on

I) heat conduction in a rod

II) diflection of vibrating string under certain conditions and to establish an expansion formula involving product of above polynomials and *H*-function of several complex variables. In the last, some interesting special cases will also be discussed. Our results are generalization of the results due to Chandel-Tiwari [1], Chaurasia-Patni [2] and Srivastava-Srivastava [16]. In special case V of problem I, it is also shown that all the results due to Chaurasia-Patni ([2],(8),(11),(12), (13), (14), (15), (16), (17), (18), (19)) are wrongly expressed. This remark also suggests that all the results due to Chaurasia and Gupta ([3], (2.1), (2.2), (3.1), (3.3), (4.1) to (4.12)) are also wrongly expressed.

8.1. Introduction. Chandel and Tiwari [1] have employed multiple hypergeometric function of several variables of Srivastava and Daoust ([7], [8], [9]; also see modified form Srivastava and Karlsson [14, p.37, eqns (2.1) to (2.3)]) in two boundary value problems.

In the present paper, first we evaluate an integral involving the product of the *H*-function of several variables of Srivastava and Panda ([10],[11], also see [12]) and several general classes of polynomials of Srivastava [6, p.1, eqn. (1.1)] defined by

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$$(8.1.1)\ S_n^{[m]}[x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s} x^s,\ n = 0,1,2,\ldots$$

where m is arbitrary positive integer and the coefficients $F_{n,s}$ (n, s^30) are arbitrary constants, real or complex.

In the last some interesting special cases will also be discussed.

Our results will be generalizations of the results due to Chandel-Tiwari [1], Chaurasia-Patni [2], and Srivastava-Srivastava [16].

8.2. Main integral. In this section, making an appeal to modified form of [4,p.372, (1)]

$$\int_0^L (\sin \pi x/L)^{W-1} \sin \pi x\lambda_m/L\, dx = \frac{W \sin \pi\lambda_m/2}{2^{W-1}\Gamma\left(\frac{W+\lambda_m+1}{2}\right)\Gamma\left(\frac{W-\lambda_m+1}{2}\right)}, \mathrm{Re}(W) > 0,$$

we evaluate the following integral very useful in our investigations:

$$(8.2.1)\ \int_0^L (\sin \pi x\ L)^{W-1} \sin(\pi x\lambda_m\ L) S_{n_1}^{m_1}\left[y_1(\sin \pi x\ L)^{2\rho_1}\right] \ldots S_{n_r}^{m_r}\left[y_r(\sin \pi x/L)^{2\rho_r}\right]$$

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} z_1(\sin \pi x/L)^{2\xi_1}, \ldots, z_n(\sin \pi x/L)^{2\xi_n}\right)dx$$

$$= \frac{L \sin(\pi\lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 s_1} A_{n_1,s_1} \cdots (-n_r)_{m_r s_r} A_{n_r,s_r}}{2^{2(\rho_1 s_1+\ldots+\rho_r s_r)}}$$

$$\frac{y_1^{s_1}}{s_1!} \cdots \frac{y_r^{s_r}}{s_r!} H(s_1,\ldots,s_r),$$

where

$$H(s_1,\ldots,s_r) = H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[1-W-2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1,\ldots,2\xi_n]:\\ [(c):\psi',\ldots,\psi^{(n)}],\left[\frac{1-W-2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}:\xi_1,\ldots,\xi_n\right]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} z_1/4^{\xi_1} \ldots z_n/4^{\xi_n}\right)$$

$Re(W) > 0$, $Re\left(W + \sum_{i=1}^{n} \xi_i d_j^{(i)}/\delta_j^{(i)}\right) > 0$, all ξ_i are real positive numbers, l, A, C, $m^{(i)}$, $n^{(i)}$, $B^{(i)}$, $D^{(i)}$ are such that $A \geq l \geq 0$, $C \geq 0$, $D^{(i)} \geq m^{(i)} \geq 0$, $B^{(i)} \geq n^{(i)} \geq 0$ and $\theta_j^{(i)}$, $j = 1, \ldots, A$; $\phi_j^{(i)}$, $j = 1, \ldots, B^{(i)}$; $\psi_j^{(i)}$, $j = 1, \ldots, C$; $\delta_j^{(i)}$, $j = 1, \ldots, D^{(i)}$ are positive real numbers,

$$\left|arg\left[z_i (\sin \pi x/L)^{2\xi_i}\right]\right| \leq \Delta_i \, \pi/2;$$

$$\Delta_i = -\sum_{j=\gamma+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0; \quad i = 1, \ldots, n.$$

Also ρ_k are real positive numbers, n_k, m_k are arbitrary positive integers, A_{n_k, s_k} are arbitrary functions of n_k and s_k real or complex independent of x, y_k, ρ_k ; $k = 1, \ldots, r$.

Problem I

8.3. Application to heat conduction in a rod. In this section, we consider a problem on outer heat conduction in a rod under certain boundary conditions. If the thermal coefficients are constants and there is no source of thermal energy, then the temperature $u(x,t)$ in one dimensional rod $0 \leq x \leq L$ satisfies the following heat equation

$$(8.3.1) \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, t \geq 0.$$

If we consider the following boundary conditions

(8.3.2) $u(0, t) = 0$,

$$(8.3.3) \quad \frac{\partial u}{\partial x}\frac{(L,t)}{} + hu(L,t) = 0, h > 0,$$

(8.3.4) $u(x, t)$ is finite as $t \to \infty$,

and initial condition

(8.3.5) $u(x, 0) = f(x)$,

then the solution of partial differential equation (8.3.1) is given by Sommerfield [5]

$$(8.3.6)\quad u(x,t)=\sum_{m=1}^{\infty}A_m\, sin(\lambda_m \pi x/L)\, exp\left\{-(\pi\lambda_m/L)^2 kt\right\},$$

where $\lambda_1, \ldots, \lambda_m$ are the roots of the transcendental equation

$$(8.3.7)\quad tan\, \pi\lambda_m = \pi\lambda_m/kL.$$

Here we consider the problem of determining $u(x,t)$, where

$$(8.3.8)\quad u(x,0) = f(x)$$

$$= (sin\, \pi x/L)^{W-1} S_{n_1}^{m_1}\left[y_1(sin\, \pi x/L)^{2\rho_1}\right] \ldots S_{n_r}^{m_r}\left[y_r(sin\, \pi x/L)^{2\rho_r}\right]$$

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\, z_1(sin\, \pi x/L)^{2\xi_1}, \ldots, z_n(sin\, \pi x/L)^{2\xi_n}\right)$$

4. Solution of the problem. Making an appeal to (8.3.6), (8.3.8) and (8.2.1) we derive

$$(8.4.1)\quad A_m' = \frac{\pi\lambda_m\, sin(\pi\lambda_m/2)}{2^{W-3}[2\pi\lambda_m - sin\, 2\pi\lambda_m]}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1 s_1} A_{n_1,s_1}\ldots(-n_r)_{m_r s_r} A_{n_r,s_r}}{4^{(\rho_1 s_1+\ldots+\rho_r s_r)}}$$

$$H(s_1,\ldots,s_r)\,\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!},$$

where all conditions of (8.2.1) are satisfied and λ_m are the roots of transcendental equation (8.3.7) and

$$H(s_1,\ldots,s_r) = H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[1-W-2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1,\ldots,2\xi_n]:\\ [(c):\psi',\ldots,\psi^{(n)}],\left[\frac{1-W-2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}:\xi_1,\ldots,\xi_n\right]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\, z_1/4^{\xi_1}\ldots z_n/4^{\xi_n}\right).$$

Now substituting the value of A_m from (8.4.1) in (8.3.6), we derive the following solution of the problem:

$$(8.4.2)\quad u(x,t) = \frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty} sin(\lambda_m \pi x/L)\, exp\left\{-(\pi\lambda_m\ L)^2 kt\right\}$$

$$\frac{\lambda_m \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 s_1} A_{n_1,s_1} \cdots (-n_r)_{m_r s_r} A_{n_r,s_r}}{4^{(\rho_1 s_1 + \ldots + \rho_r s_r)}}$$

$$H(s_1,\ldots,s_r) \frac{y_1^{s_1}}{s_1!} \cdots \frac{y_r^{s_r}}{s_r!},$$

where all conditions of (8.4.1) are satisfied.

8.5. Expansion Formula. Making an appeal to (8.3.8) and (8.4.2), we derive the expansion formula :

$$(8.5.1) \quad (\sin \pi x/L)^{W-1} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left(y_1(\sin \pi x/L)^{2\rho_1},\ldots,y_r(\sin \pi x/L)^{2\rho_r}\right)$$

$$H_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} z_1(\sin \pi x/L)^{2\xi_1},\ldots,z_n(\sin \pi x/L)^{2\xi_n}\right)$$

$$= \frac{\pi}{2^{W-3}} \sum_{m=1}^{\infty} \sin(\pi\lambda_m x\ L) \frac{\lambda_m \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 s_1} A_{n_1,s_1} \cdots (-n_r)_{m_r s_r} A_{n_r,s_r}}{4^{(\rho_1 s_1 + \ldots + \rho_r s_r)}}$$

$$H[s_1,\ldots,s_r] \frac{y_1^{s_1}}{s_1!} \cdots \frac{y_r^{s_r}}{s_r!},$$

provided that all the conditions of (8.4.1) are satisfied.

Problem II

8.6. Application to Homogeneous Wave Problem.

In this section, we shall determine the shape (deflection) $u(x,t)$ of vibrating string. If the deflection due to weight of string is negligible (usually the case), then $u(x,t)$ satisfies the partial differential equation:

$$(8.6.1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c} \frac{\partial^2 u}{\partial t^2}, \qquad 0<x<L,\ t>0.$$

If we assume the boundary conditions

(8.6.2) $u\,(0,t)=0,\qquad u\,(L,\ t)=0,\qquad t>0$

and initial conditions

$$(8.6.3)\quad \frac{\partial u(x,0)}{\partial t}=g(x)\ \text{(initial velocity)}$$

and

(8.6.4) $u\,(x,0)=f\,(x)$.

Then the solution of partial differential equation (8.6.1) is given by

$$(8.6.5)\quad u(x,t)=\sum_{m=1}^{\infty}\left[A_m\cos(\pi\lambda_m ct\ L)+B_m\sin(\pi\lambda_m ct\ L)\right]\sin(\pi\lambda_m x/L).$$

Now we consider the problem of determining $u(x,t)$, where $u(x,0)=f(x)$ is given by (8.3.8) and

$$(8.6.6)\quad g(x)=(\sin\pi x\ L)^{W'-1}S_{p_1}^{q_1}\left[x_1(\sin\pi x\ L)^{2\sigma_1}\right]\ldots S_{p_l}^{q_l}\left[x_l(\sin\pi x/L)^{2\sigma_l}\right]$$

$$H_{V,W:[P',Q'];\ldots;[P^{(n)},Q^{(n)}]}^{0,u:(M',N');\ldots;(M^{(n)},N^{(n)})}\left(\begin{matrix}[(e):E',\ldots,E^{(n)}]:[(f'):F'];\ldots;[(f^{(n)}):F^{(n)}];\\ [(g):G',\ldots,G^{(n)}]:[(h'):H'];\ldots;[(h^{(n)}):H^{(n)}];\end{matrix}\right.$$

$$\left.u_1\sin^{2\eta_1}\pi x/L,\ldots,u_n\sin^{2\eta_n}\pi x/L\right).$$

Now making an appeal to (8.6.4) and (8.6.5), we obtain

$$(8.6.7)\quad u(x,0)=f(x)=\sum_{m=1}^{\infty}A_m\ \sin(\pi x\lambda_m/L),$$

while an appeal to (8.6.3) and (8.6.5) gives

$$(8.6.8)\quad \frac{\partial(u(x,0))}{\partial t}=g(x)=\frac{\pi c}{L}\sum_{m=1}^{\infty}B_m\lambda_m\sin(\pi\lambda_m x\ L).$$

Now making an appeal to (8.2.1), (8.3.8) and (8. 6.4), A_m is given by (8.4.1).

Again by (8.6.6), (8.6.8) and (8.2.1), we also derive

$$(8.6.9)\ B_m=\frac{L\sin(\pi\lambda_m/2)}{2^{W'-3}c[2\pi\lambda_m-\sin 2\pi\lambda_m]}\sum_{s_1=0}^{[p_1/q_1]}\cdots\sum_{s_l=0}^{[p_l/q_l]}\frac{(-p_1)_{q_1s_1}A_{p_1,s_1}\cdots(-p_l)_{q_ls_l}A_{p_l,s_l}}{2^{2(\sigma_1s_1+\ldots+\sigma_ls_l)}}$$

$$\overline{H}[s_1,\ldots,s_l]\frac{x_1^{s_1}}{s_1!}\cdots\frac{x_l^{s_l}}{s_l!},$$

where

$$\overline{H}[s_1,\ldots,s_1] = H^{0,u+1:(M',N');\ldots;(M^{(n)},N^{(n)})}_{v+1,w+2:[P',Q'];\ldots;[P^{(n)},Q^{(n)}]}\left(\begin{matrix}[(e):E',\ldots,E^{(n)}],\\ [(g):G',\ldots,G^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-W'-2(\sigma_1 s_1+\ldots+\sigma_l s_l):2\eta_1,\ldots,2\eta_n]: & [(f'):F'];\ldots;[(f^{(n)}):F^{(n)}];\\ \left[\dfrac{1-W'-2(\sigma_1 s_1+\ldots+\sigma_l s_l)\pm\lambda_m}{2}:\eta_1,\ldots,\eta_n\right]: & [(h'):H'];\ldots;[(h^{(n)}):H^{(n)}];\end{matrix}\ \frac{u_1}{4^{\eta_1}};\ldots;\frac{u_n}{4^{\eta_n}}\right).$$

Now substituting the values of A_m and B_m in (8.6.5), the solution of the problem in given by

$$u(x,t)=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\frac{\lambda_m\, sin(\pi\lambda_m x/L)\, cos(\pi\lambda_m ct/L)\, sin(\pi\lambda_m/2)}{[2\pi\lambda_m - sin\, 2\pi\lambda_m]} \tag{8.6.10}$$

$$\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1 s_1}A_{n_1 s_1}\cdots(-n_r)_{m_r s_r}A_{n_r s_r}}{2^{2(\rho_1 s_1+\ldots+\rho_r s_r)}}H(s_1,\ldots,s_r)\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}+\frac{\pi L}{c.2^{W'-3}}$$

$$\sum_{m=1}^{\infty}\frac{sin(\pi\lambda_m x/L)\, sin(\pi\lambda_m ct/L)\, sin(\pi\lambda_m/2)}{[2\pi\lambda_m - sin\, 2\pi\lambda_m]}$$

$$\sum_{s_1=0}^{[p_1/q_1]}\cdots\sum_{s_k=0}^{[p_k/q_k]}\frac{(-p_1)_{q_1 s_1}A_{p_1,s_1}\cdots(-p_l)_{q_l s_l}A_{p_l,s_l}}{2^{2(\sigma_1 s_1+\ldots+\sigma_l s_l)}}H(s_1,\ldots s_l)\frac{x_1^{s_1}}{s_1!}\cdots\frac{x_l^{s_l}}{s_l!},$$

where l_m are the roots of equation (8.3.7), $Re(w)>0$, $Re(W')>0$, $Re\left(W'+\sum_{i=1}^{n}\eta_i\, h_j^{(i)}\ H_j^{(i)}\right)>0$, all x_i, h_i are positive real numbers and $u,v,w,$ $N^{(i)},M^{(i)},P^{(i)}$, $Q^{(i)}$ are such that $v\geq u\geq 0$, $w\geq 0$, $Q^{(i)}\geq M^{(i)}\geq 0$, $P^{(i)}\geq N^{(i)}\geq 0$ and $E_j^{(i)}, j=1,\ldots,v$; $F_j^{(i)}, j=1,\ldots,P^{(i)}$; $G_j^{(i)}, j=1,\ldots,w$; $H_j^{(i)}, j=1,\ldots,Q^{(i)}$ are positive real numbers;

$$\left|arg\left(y_i\, sin^{2\eta_i}\, \pi x/L\right)\right| < T_i\, \pi/2\,,$$

where

$$T_i = -\sum_{j=u+1}^{v} E_j^{(i)} + \sum_{j=1}^{N^{(i)}} F_j^{(i)} - \sum_{j=1+N^{(i)}}^{P^{(i)}} F_j^{(i)} - \sum_{j=1}^{w} G_j^{(i)} + \sum_{j=1}^{M^{(i)}} H_j^{(i)} - \sum_{j=1+M^{(i)}}^{Q^{(i)}} H_j^{(i)} > 0;\ i=1,\ldots,n.$$

p_r, q_r are arbitrary positive integers, A_{p_r,s_r} are arbitrary functions of p_r, s_r real or complex independent of x, x_r, s_r; $r=1,\ldots,k$ and also all conditions of (8.2.1) are satisfied.

8.8. Special Cases of Problem 1.

Case I. For each $m_i = 2, A_{n_i,s_i} = (-1)^{s_i}, S_{n_i}^2(x) \to x^{n_i} H_{n_i}\left(\dfrac{1}{2\sqrt{x}}\right)$, we derive the result from (8.2.1) for Hermite polynomials ([15] p.106, eq. (5.5.4) and [13],p.158)

$$(8.8.1)\ \int_0^L (\sin \pi x\ L)^{W-1} \sin(\pi x\lambda_m\ L)\left[y_1(\sin\pi x\ L)^{2\rho_1}\right]^{n_1/2} H_{n_1}\left(\frac{1}{2\sqrt{y_1(\sin\pi x/L)^{2\rho_1}}}\right)$$

$$\ldots\left[y_r(sin\,\pi x\ L)^{2\rho_r}\right]^{n_r/2} H_{n_r}\left(\frac{1}{2\sqrt{y_r(sin\,\pi x/L)^{2\rho_r}}}\right) H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1(sin\,\pi x/L)^{2\xi_1},\ldots,z_n(sin\,\pi x/L)^{2\xi_n}\right)dx$$

$$= \frac{L\, sin(\pi\lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \frac{(-n_1)_{2s_1}}{s_1!} \cdots \frac{(-n_r)_{2s_r}}{s_r!} \frac{(-1)^{s_1+\ldots+s_r} y_1^{s_1}\ldots y_r^{s_r}}{2^{2(\rho_1 s_1+\ldots+\rho_r s_r)}} H(s_1,\ldots,s_r)$$

valid if all conditions of (8.2.1) hold.

Then by (8.4.2), the solution of problem I, is given by

$$(8.8.2)\ u(x,t) = \frac{\pi}{2^{W-3}} \sum_{m=1}^{\infty} sin(\lambda_m \pi x/L)\, exp\left\{-(\pi\lambda_m\ L)^2 kt\right\}$$

$$\frac{\lambda_m \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} \sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \frac{(-n_1)_{2s_1} \cdots (-n_r)_{2s_r} (-1)^{s_1+\ldots+s_r}}{2^{2(\rho_1 s_1+\ldots+\rho_r s_r)}} \frac{y_1^{s_1} \ldots y_r^{s_r}}{s_1! \ldots s_r!} H(s_1,\ldots,s_r)$$

where all conditions of (8.4.2) are satisfied.
From (8.5.1), we derive expansion formula

$$(8.8.3) \quad (\sin \pi x\ L)^{W-1} y_1^{n_1/2} \ldots y_r^{n_r/2} (\sin \pi x/L)^{n_1\rho_1+\ldots+n_r\rho_r} H_{n_1}\left(\frac{1}{2\sqrt{y_1}(\sin \pi x/L)^{\rho_1}}\right)$$

$$\ldots H_{n_r}\left(\frac{1}{2\sqrt{y_r}(\sin \pi x/L)^{\rho_r}}\right) H^{0,\lambda:(\mu',v');\ldots;(\mu^{(n)},v^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} z_1(\sin \pi x/L)^{2\xi_1},\ldots,z_n(\sin \pi x/L)^{2\xi_n}\right)$$

$$= \frac{\pi}{2^{W-3}} \sum_{m=1}^{\infty} \frac{\sin(\pi\lambda_m x/L)\lambda_m \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} \sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \frac{(-n_1)_{2s_1} \cdots (-n_r)_{2s_r} (-1)^{s_1+\ldots+s_r}}{2^{2(\rho_1 s_1+\ldots+\rho_r s_r)}}$$

$$H[s_1,\ldots,s_r] \frac{y_1^{s_1}}{s_1!} \cdots \frac{y_r^{s_r}}{s_r!}.$$

Case II. For each $m_i = 1, A_{n_i s_i} = \dfrac{(1+\alpha_i)_{n_i}}{n_i!} \dfrac{1}{(1+\alpha_i)_{s_i}}, i = 1,\ldots,r;$ we derive the result for laguerre polynomials $S_{n_i}^1(x) \to L_{n_i}^{(\alpha_i)}$ ([15], p. 101, Eq. (5.1.6) and [13], p.159)

(8.8.4)

$$\int_0^L (\sin \pi x\ L)^{W-1} \sin(\pi x\lambda_m\ L) L_{n_1}^{(\alpha_1)}\left[y_1(\sin \pi x\ L)^{2\rho_1}\right] \ldots L_{n_r}^{(\alpha_r)}\left[y_r(\sin \pi x/L)^{2\rho_r}\right]$$

$$H^{0,\lambda:(\mu',v');\ldots;(\mu^{(n)},v^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} z_1(\sin \pi x/L)^{2\xi_1},\ldots,z_n(\sin \pi x/L)^{2\xi_n}\right) dx$$

$$= \frac{L\,sin(\pi\lambda_m/2)}{2^{W-1}} \frac{(1+\alpha_i)_{n_1}}{n_1!} \cdots \frac{(1+\alpha_r)_{n_r}}{n_r!} \sum_{s_1=0}^{n_1} \cdots \sum_{s_r=0}^{n_r} \frac{(-n_1)_{s_1} \cdots (-n_r)_{s_r}}{(1+\alpha_1)_{s_1} \cdots (1+\alpha_r)_{s_r}}$$

$$\frac{(y_1/4^{\rho_1})^{s_1}}{s_1!} \cdots \frac{(y_r/4^{\rho_r})^{s_r}}{s_r!} H[s_1,\ldots,s_r],$$

where all condition of (8.2.1) are satisfied.

Now by (8.4.2) the solution of peoblem I, reduces to

$$(8.8.5)\quad u(x,t) = \frac{\pi}{2^{W-3}} \sum_{m=1}^{\infty} sin(\lambda_m \pi x/L)\, exp\{-(\pi\lambda_m/L)^2 kt\}$$

$$\frac{\lambda_m\, sin(\pi\lambda_m/2)}{[2\pi\lambda_m - sin\, 2\pi\lambda_m]} \frac{(1+\alpha_1)_{n_1}}{n_1!} \cdots \frac{(1+\alpha_r)_{n_r}}{n_r!} \sum_{s_1=0}^{n_1} \cdots \sum_{s_r=0}^{n_r} \frac{(-n_1)_{s_1}}{(1+\alpha_1)_{s_1}} \cdots \frac{(-n_r)_{s_r}}{(1+\alpha_r)_{s_r}}$$

$$H[s_1,\ldots,s_r] \frac{y_1^{s_1}}{s_1!} \cdots \frac{y_r^{s_r}}{s_r!} \frac{1}{4^{(\rho_1 s_1+\ldots+\rho_r s_r)}},$$

where all conditions of (8.4.1) are satisfied.

Also from (8.5.1) we derive expansion formula

$$(8.8.6)\quad (sin\,\pi x\, L)^{W-1} L_n^{(\alpha_1)}\left[y_1 (sin\,\pi x\, L)^{2\rho_1}\right] \ldots L_{n_r}^{(\alpha_r)}\left[y_r (sin\,\pi x/L)^{2\rho_r}\right]$$

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1 (sin\,\pi x/L)^{2\xi_1},\ldots,z_n (sin\,\pi x/L)^{2\xi_n}\right)$$

$$= \frac{\pi}{2^{W-3}} \sum_{m=1}^{\infty} sin(\pi\lambda_m x\, L) \frac{\lambda_m\, sin(\pi\lambda_m/2)}{[2\pi\lambda_m - sin\, 2\pi\lambda_m]} \frac{(1+\alpha_1)_{n_1}}{n_1!} \cdots \frac{(1+\alpha_r)_{n_r}}{n_r!}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \frac{(-n_1)_{s_1} \cdots (-n_r)_{s_r}}{(1+\alpha_1)_{s_1} \cdots (1+\alpha_r)_{s_r}} \frac{(4^{-\rho_1} y_1)^{s_1}}{s_1!} \cdots \frac{(4^{-\rho_r} y_r)^{s_r}}{s_r!} H(s_1,\ldots,s_r),$$

where all the conditions of (8.5.1) are satisfied

Case III. For each $m_i=1,\ q_i=1,\ A_{n_i s_i}=\dfrac{(1+\alpha_i)_{n_i}}{n_i!}\dfrac{(1+\alpha_i+\beta_i+n_i)}{(1+\alpha_i)_{s_i}},S^1_{n_i}(x)\to P^{(\alpha_i,\beta_i)}_{n_i}$,

$i=1,2,\ldots,r$ we derive the result from (8.2.1) for Jacobi polynomials ([15], p.68, Eq. (4.3.2) and [13], p. 159)

$$(8.8.7)\ \int_0^L (\sin\pi x\ L)^{W-1}\sin(\pi x\lambda_m\ L)\prod_{i=1}^{r}P^{(\alpha_i,\beta_i)}_{n_i}\left[1-2y_i(\sin\pi x\ L)^{2\rho_i}\right]$$

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.$$

$$\left. z_1(\sin\pi x/L)^{2\xi_1},\ldots,z_n(\sin\pi x/L)^{2\xi_n}\right)dx$$

$$=\frac{L\sin(\pi\lambda_m/2)}{2^{W-1}}\prod_{i=1}^{r}\frac{(1+\alpha_i)_{n_i}}{n_i!}\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\frac{(-n_1)_{s_1}\cdots(-n_r)_{s_r}\,y_1^{s_1}\cdots y_r^{s_r}}{2^{2(\rho_1 s_1+\ldots+\rho_r s_r)}\,s_1!\ldots s_r!}$$

$$\prod_{i=1}^{r}\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}H[s_1,\ldots,s_r]$$

valid if all the conditions of (8.2.1) are satisfied.

Then by (8.4.2), the solution of the problem I is given by

$$(8.8.8)\ u(x,t)=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\sin(\pi x/L)\exp\{-(\pi\lambda_m/L)^2kt\}\ \frac{\lambda_m\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin2\pi\lambda_m]}$$

$$\prod_{i=1}^{r}\frac{(1+\alpha_i)}{n_i!}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{s_i}}{4^{\rho_i s_i}}\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}\frac{y_i^{s_i}}{s_i!}\ H(s_1,\ldots,s_r)$$

where all conditions (8.2.1) and (8.4.2) are astified.

Also expanion formula (8.5.1), reduces to

$$(8.8.9)\ \ (\sin\pi x\ L)^{W-1}\prod_{i=1}^{r}P^{(\alpha_i,\beta_i)}_{n_i}\left(1-2y_i(\sin\pi x\ L)^{\rho_i}\right)H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]; \\ [(d'):\delta'];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1(\sin \pi x/L)^{2\xi_1},\ldots,z_n(\sin\pi x/L)^{2\xi_n}\right)$$

$$= \frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\frac{\sin(\pi\lambda_m x/L)\,\lambda_m \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]}\prod_{i=1}^{r}\frac{(1+\alpha_i)_{n_i}}{n_i\,!}\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\prod_{i=1}^{r}\frac{(-n_i)_{s_i}}{4^{\rho_i s_i}}$$

$$\frac{(1+\alpha_i+\beta_i+n_i)_{s_1}}{(1+\alpha_i)_{s_i}}\frac{y_i^{s_i}}{s_i\,!}H(s_1,\ldots,s_r)\,,$$

valid if all the conditions of (8.2.1) and (8.4.2) are satisfied.

Case IV. Choosing $l=A$, $\mu^{(i)}=1, \nu^{(i)}=B^{(i)}$, replacing $D^{(i)}$ by $D^{(i)}+1$, $i=1,\ldots,n$, integral (8.2.1) reduces to

$$(8.8.10)\quad \int_0^L(\sin\pi x/L)^{W-1}\sin(\pi x\lambda_m/L)\prod_{i=1}^{r}S_{n_i}^{m_i}y_i(\sin\pi x/L)^{2\rho_i}\;F^{A:B';\ldots;B^{(n)}}_{C,D';\ldots;D^{(n)}}$$

$$\left(\begin{matrix}\left[1-(a):\theta',\ldots,\theta^{(n)}\right]:[1-(b'):\phi'];\ldots;\left[1-\left(b^{(n)}\right):\phi^{(n)}\right]; \\ \left[1-(c):\psi',\ldots,\psi^{(n)}\right]:[1-(d'):\delta'];\ldots;\left[1-\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1(\sin\pi x/L)^{2\xi_1},\ldots,z_n(\sin\pi x/L)^{2\xi_n}\right)dx$$

$$= \frac{L\sin(\pi\lambda_m/2)}{2^{W-1}}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i s_i}}{4^{\rho_i s_i}}\frac{A_{n_i,s_i}}{s_i\,!}\,y_i^{s_i}\;F^{A+1:B';\ldots;B^{(n)}}_{C+2,:D';\ldots;D^{(n)}}$$

$$\left(\begin{matrix}\left[1-(a):\theta',\ldots,\theta^{(n)}\right],\left[W+2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1,\ldots,2\xi_n\right]: & [1-(b'):\phi'];\ldots;\left[1-\left(b^{(n)}\right):\phi^{(n)}\right]; \\ \left[1-(c):\psi',\ldots,\psi^{(n)}\right],\left[\frac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}:\xi_1,\ldots,\xi_n\right]: & [1-(d'):\delta'];\ldots;\left[1-\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1/4^{\xi_1}\ldots z_n/4^{\xi_n}\right)$$

$$\frac{\Gamma(W+2(\rho_1 s_1+\ldots+\rho_r s_r))}{\Gamma\left(\frac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}\right)},$$

where all conditions of (8.2.1) are satisfied.

Then the solution (8.4.2) of problem I, is given by

$$(8.8.11)\quad u(x,t)=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty} sin(\lambda_m \pi x/L)\, exp\left\{-(\pi\lambda_m/L)^2 kt\right\}\ \frac{\lambda_m\, sin(\pi\lambda_m/2)}{[2\pi\lambda_m - sin\, 2\pi\lambda_m]}$$

$$\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i s_i}}{4^{\rho_i s_i}}\frac{A_{n_i,s_i}\,y_i^{s_i}}{s_i\,!}\ \frac{\Gamma(W+2(\rho_1 s_1+\ldots+\rho_r s_r))}{\Gamma\left(\dfrac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}\right)}$$

$$\cdot\ F^{A+1:B';\ldots;B^{(n)}}_{C+2,:D';\ldots;D^{(n)}}\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(n)}],[W+2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1,\ldots,2\xi_n]:\\ [1-(c):\psi',\ldots,\psi^{(n)}],\left[\dfrac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}:\xi_1,\ldots,\xi_n\right]:\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\ldots;\left[1-(b^{(n)}):\phi^{(n)}\right];\\ [1-(d'):\delta'];\ldots;\left[1-(d^{(n)}):\delta^{(n)}\right];\end{matrix}\ z_1/4^{\xi_1},\ldots,z_n/4^{\xi_n}\right)$$

provided that all conditions of (8.2.1) and (8.4.2) are satisfied.

Also expansion formula (8.5.1) reduces to

$$(8.8.12)\quad (sin\,\pi x\ L)^{W-1} S_{n_1}^{m_1}\left[y_1(sin\,\pi x\ L)^{2\rho_1}\right]\ldots S_{n_r}^{m_r}\left[y_r(sin\,\pi x/L)^{2\rho_r}\right]\ F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(n)}]:[1-(b'):\phi'];\ldots;\left[1-(b^{(n)}):\phi^{(n)}\right];\\ [1-(c):\psi',\ldots,\psi^{(n)}]:[1-(d'):\delta'];\ldots;\left[1-(d^{(n)}):\delta^{(n)}\right];\end{matrix}\ z_1(sin\,\pi x/L)^{2\xi_1},\ldots,z_n(sin\,\pi x/L)^{2\xi_n}\right)dx$$

$$=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\frac{sin(\pi\lambda_m x/L)\ \ \lambda_m\, sin(\pi\lambda_m/2)}{[2\pi\lambda_m - sin\,2\pi\lambda_m]}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i s_i}}{4^{\rho_i s_i}}\frac{A_{n_i,s_i}\,y_i^{s_i}}{s_i\,!}$$

$$\frac{\Gamma(W+2(\rho_1 s_1+\ldots+\rho_r s_r))}{\Gamma\left(\dfrac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}\right)}\ F^{A+1:B';\ldots;B^{(n)}}_{C+2,:D';\ldots;D^{(n)}}$$

$$\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(n)}],[W+2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1,\ldots,2\xi_n]:\\ [1-(c):\psi',\ldots,\psi^{(n)}],\left[\dfrac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}:\xi_1,\ldots,\xi_n\right]:\end{matrix}\ \begin{matrix}[1-(b'):\phi'];\ldots;\left[1-(b^{(n)}):\phi^{(n)}\right];\\ [1-(d'):\delta'];\ldots;\left[1-(d^{(n)}):\delta^{(n)}\right];\end{matrix}\ z_1/4^{\xi_1}\ldots z_n/4^{\xi_n}\right)$$

where all the conditions of (8.2.1) and (8.4.2) are satisfied.

Case V. Further for $r=2$, $\lambda_m = (2m+1)/2$, our results (8.2.1), (8.3.8), (8.4.1), (8.4.2), (8.8.1), (8.8.2), (8.8.4), (8.8.5), (8.8.7), (8.8.8), (8.8.10) and (8.8.11) give respectively (7), (6) and correct forms of wrong results (11), (8), (12), (13), (14), (15), (16), (17), (18) and (19) due to Chaurasia and Patni [2].

This remark also suggests that all the results due to Chaurasia and Gutpa ([3]), (2.1), (2.2), (3.1), (3.3), (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10), (4.11) and (4.12) are wrongly expressed. Actually $2^{-2(\rho s+\rho' s')}$ should be written within the summation signs $\sum_{s=0}^{n} \sum_{s'=0}^{n'}$ in the all above results.

Case VI. Choosing $l=A$, $n_i=0$, $\mu^{(i)} = 1, \nu^{(i)} = B^{(i)}$, replacing $D^{(i)}$ by $D^{(i)}+1$, z_i by $-z_i$ $i=1,\ldots,n$, all the results of this paper will reduce to the results of Chandel and Tiwari [1]

Similarly specializing the coefficents A_{n_i,s_i} $i=1,\ldots,n$ of polynomials $S_{n_i}^{m_i}[x_i]$ [6] and parameters of H-function of several complex variables ([10], [11], [12]), we shall get a large number of results involving various polynomials and other special functions useful in Mathmatical Analysis, Applied Mathematics and Mathematical Physics.

9. Special Cases of Problem II

Case I. For Hermite polynomials ([15], p.106, eq. (5.54), and [13], p.158), choosing each $m_i = 2, A_{n_i,s_i} = (-1)^{s_i}, S_{n_i}^2(x) \to x^{n_i} H_{n_i}\left(\frac{1}{2\sqrt{x}}\right), i = 1,\ldots,r;$

$q_j = 2,\ j = 1,\ldots,l$ (8.3.8) and (8.6.6) give respectively :

$$(8.9.1)\quad f(x) = (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} y_i (\sin \pi x/L)^{2\rho_i} H_{n_i}\left(\frac{1}{2\sqrt{y_i}(\sin \pi x/L)^{\rho_i}}\right)$$

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1(\sin \pi x/L)^{2\xi_1},\ldots,z_n(\sin \pi x/L)^{2\xi_n}\right)$$

and

$$(8.9.2)\ g(x)=(\sin \pi x/L)^{W'-1}\prod_{i=1}^{l} x_i(\sin \pi x/L)^{2\sigma_i} H_{p_i}\left(\frac{1}{2\sqrt{x_i}(\sin \pi x/L)^{\sigma_i}}\right)$$

$$H^{0,u:(M',N');\ldots;(M^{(n)},N^{(n)})}_{v,w:[P',Q'];\ldots;[P^{(n)},Q^{(n)}]}\left(\begin{matrix}[(e):E',\ldots,E^{(n)}]:[(f'):F'];\ldots;[(f^{(n)}):F^{(n)}];\\ [(g):G',\ldots,G^{(n)}]:[(h'):H'];\ldots;[(h^{(n)}):H^{(n)}];\end{matrix}\right.$$

$$\left. u_1 \sin^{2\eta_1}\pi x/L,\ldots,u_n \sin^{2\eta_n}\pi x/L\right)$$

Then solution (8.6.10) of the problem II is reduced in the form:

$$(8.9.3)\quad u(x,t)=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\frac{\lambda_m \sin(\pi\lambda_m x/L)\cos(\pi\lambda_m ct/L)\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}$$

$$\sum_{s_1=0}^{[n_1/2]}\cdots\sum_{s_r=0}^{[n_r/2]}\prod_{i=1}^{r}\frac{(-1)^{s_i}(-n_i)_{2s_i}\,y^{s_i}}{4^{2\rho_i s_i}\,s_i\,!}H(s_1,\ldots,s_r)$$

$$+\frac{\pi L}{c.2^{W'-3}}\sum_{m=1}^{\infty}\frac{\sin(\pi\lambda_m x/L)\sin(\pi\lambda_m ct/L)\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}$$

$$\sum_{s_1=0}^{[p_1/2]}\cdots\sum_{s_l=0}^{[p_l/2]}\prod_{i=1}^{l}\frac{(-1)^{s_i}(-p_i)_{2s_i}\,x^{s_i}}{4^{2\rho_i s_i}\,s_i\,!}\overline{H}(s_1,\ldots,s_l)$$

where all conditins of (8.6.10) are satisfied.

Case II. For Laguerre polynomials ([15], p.101, Eq. 5. 1.6) and [13], p.159), choosing each $m_i=1, A_{n_i,s_i}=\frac{(1+\alpha_i)_{n_i}}{n_i\,!}\frac{1}{(1+\alpha_i)_{s_i}}, S^1_{n_i}(x)\to L^{(\alpha_i)}_{n_i}, i=1,\ldots,r;\ q_j=1,$

$A_{p_j,s_j}=\frac{(1+\beta_j)_{p_j}}{p_j\,!}\frac{1}{(1+\beta_j)_{s_j}}, S^1_{p_j}(x)\to L^{(\beta_j)}_{p_j}, j=1,\ldots,l,$ (8.3.8) and (8.4.6) give

respectively

$$(8.9.4) \quad f(x) = (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} L_{n_i}^{(\alpha_i)}\left[y_i (\sin \pi x/L)^{2\rho_i} \right] H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} z_1 (\sin \pi x/L)^{2\xi_1}, \ldots, z_n (\sin \pi x/L)^{2\xi_n} \right)$$

and

$$(8.9.5) \quad g(x) = (\sin \pi x \ L)^{W'-1} \prod_{i=1}^{l} L_{p_i}^{(1+\beta_i)} \left[x_i (\sin \pi x \ L)^{2\sigma_i} \right]$$

$$H^{0,u:(M',N');\ldots;(M^{(n)},N^{(n)})}_{v,w:[P',Q'];\ldots;[P^{(n)},Q^{(n)}]} \left(\begin{matrix} [(e):E',\ldots,E^{(n)}]:[(f'):F'];\ldots;[(f^{(n)}):F^{(n)}]; \\ [(g):G',\ldots,G^{(n)}]:[(h'):H'];\ldots;[(h^{(n)}):H^{(n)}]; \end{matrix} \right.$$

$$\left. u_1 \sin^{2\eta_1} \pi x/L, \ldots, u_n \sin^{2\eta_n} \pi x/L \right)$$

Then solution (8.6.10) of the problem is reduced to

$$(8.9.6) \quad u(x,t) = \frac{\pi}{2^{W-3}} \sum_{m=1}^{\infty} \frac{\lambda_m \sin(\pi\lambda_m x/L) \cos(\pi\lambda_m ct/L) \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]}$$

$$\prod_{i=1}^{r} \frac{(1+\alpha_i)_{n_i}}{n_i!} \sum_{s_1=0}^{n_1} \cdots \sum_{s_r=0}^{n_r} \prod_{i=1}^{r} \frac{(-n_i)_{s_i}}{(1+\alpha_i)_{s_i}} \frac{y^{s_i}}{4^{2\rho_i s_i} s_i!} H(s_1,\ldots,s_r)$$

$$+ \frac{\pi L}{c.2^{W'-3}} \sum_{m=1}^{\infty} \frac{\sin(\pi\lambda_m x/L) \sin(\pi\lambda_m ct/L) \sin(\pi\lambda_m/2)}{[2\pi\lambda_m - \sin 2\pi\lambda_m]}$$

$$\prod_{j=1}^{l} \frac{(1+\beta_j)_{p_j}}{p_j!} \sum_{s_1=0}^{p_1} \cdots \sum_{s_l=0}^{p_l} \prod_{i=1}^{l} \frac{(-p_i)_{s_i}}{(1+\beta_i)_{s_i}} \frac{x^{s_i}}{4^{\sigma_i s_i} s_i!} H(s_1,\ldots,s_l).$$

where all conditions of (8.6.10) are satisfied

Case III For each $m_i=1,\ q_i=1,\ A_{n_i,s_i} = \dfrac{(1+\alpha_i)_{n_i}}{n_i!} \dfrac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}$,

$$A_{p_i,s_i}=\frac{(1+\gamma_i)_{p_i}}{p_i!}\frac{(1+\gamma_i+\delta_i+p_i)_{s_i}}{(1+\gamma_i)_{s_i}},S^1_{n_i}[x]\rightarrow P^{(\alpha_i,\beta_i)}_{n_i}(1-2x),S^1_{p_i}[x]\rightarrow P^{(\gamma_i,\delta_i)}_{p_i}(1-2x)$$

we derive for Jacobi polynomials ([15], p.68, eq. (4.32) and [13], p. 159) from the results (8.3.8) and (8.6.6) respectively

$$(8.9.7)\quad f(x)=(\sin\pi x/L)^{W-1}\prod_{i=1}^{r}P^{(\alpha_i,\beta_i)}_{n_i}\left[1-2y_i(\sin\pi x/L)^{2\rho_i}\right]H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}z_1(\sin\pi x/L)^{2\xi_1},\ldots,z_n(\sin\pi x/L)^{2\xi_n}\right)$$

and

$$(8.9.8)\,g(x)=(\sin\pi x\ L)^{W'-1}\prod_{i=1}^{l}P^{(\gamma_i,\delta_i)}_{p_i}\left[1-2x_i(\sin\pi x\ L)^{2\sigma_i}\right]$$

$$H^{0,u:(M',N');\ldots;(M^{(n)},N^{(n)})}_{v,w:[P',Q'];\ldots;[P^{(n)},Q^{(n)}]}\left(\begin{matrix}[(e):E',\ldots,E^{(n)}]:[(f'):F'];\ldots;[(f^{(n)}):F^{(n)}];\\ [(g):G',\ldots,G^{(n)}]:[(h'):H'];\ldots;[(h^{(n)}):H^{(n)}];\end{matrix}\right.$$

$$\left.u_1\sin^{2\eta_1}\pi x/L,\ldots,u_n\sin^{2\eta_n}\pi x/L\right).$$

The solution of the problem II, reduces to

$$(8.9.9)\quad u(x,t)=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\frac{\lambda_m\sin(\pi\lambda_m x/L)\cos(\pi\lambda_m ct/L)\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}$$

$$\prod_{i=1}^{r}\frac{(1+\alpha_i)_{n_i}}{n_i!}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{s_i}(1+\alpha_i+\beta_i+n_i)_{s_i}y_i^{s_i}}{(1+\alpha_i)_{s_i}\ 4^{\rho_i s_i}s_i!}H(s_1,\ldots,s_r)$$

$$+\frac{\pi L}{c.2^{W'-3}}\frac{\sin(\pi\lambda_m x/L)\sin(\pi\lambda_m ct/L)\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}$$

$$\prod_{i=1}^{l}\frac{(1+\gamma_i)_{p_i}}{p_i!}\sum_{s_1=0}^{p_1}\cdots\sum_{s_l=0}^{p_l}\prod_{i=1}^{l}\frac{(-p_i)_{s_i}(1+\gamma_i+\delta_i+p_i)_{s_i}x_i^{s_i}}{(1+\gamma_i)_{n_i}\ 4^{\sigma_i s_i}\ s_i!}H(s_1,\ldots,s_l)$$

where all conditions of (8.6.10) ware satisfied.

Case IV. Choosing l=A, $\mu^{(i)}=1, \nu^{(i)}=B^{(i)}$, replacing $D^{(i)}$ by $1-D^{(i)}$, z_i by $(-z_i)$ in (8.3.8); and Choosing $u=v$, $M^{(i)}=1$, $N^{(i)}=P^{(i)}$ replacing by $Q^{(i)}$ by $1-Q^{(i)}$, u_i by $(-u_i)$, $i=1,\ldots,n$ in (8.6.6), we derive respectively

$$(8.9.10)\quad f(x)=(\sin \pi x/L)^{W-1}\prod_{i=1}^{r} S_{n_i}^{m_i}\left[y_i(\sin \pi x/L)^{2\rho_i}\right] F^{A:B';\ldots;B^{(n)}}_{C,D';\ldots;D^{(n)}}$$

$$\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(n)}]:[1-(b'):\phi'];\ldots;[1-(b^{(n)}):\phi^{(n)}];\\ [1-(c):\psi',\ldots,\psi^{(n)}]:[1-(d'):\delta'];\ldots;[1-(d^{(n)}):\delta^{(n)}];\end{matrix} z_1(\sin \pi x/L)^{2\xi_1},\ldots,z_n(\sin \pi x/L)^{2\xi_n}\right)$$

and

$$(8.9.11)\quad g(x)=(\sin \pi x\ L)^{W'-1}\prod_{i=1}^{l} S_{p_i}^{q_i}\left[x_i(\sin \pi x\ L)^{2\sigma_i}\right]$$

$$F^{\upsilon:P';\ldots;P^{(n)}}_{W:Q';\ldots;Q^{(n)}}\left(\begin{matrix}[1-(e):E',\ldots,E^{(n)}]:[1-(f'):F'];\ldots;[1-(f^{(n)}):F^{(n)}];\\ [1-(g):G',\ldots,G^{(n)}]:[1-(h'):H'];\ldots;[1-(h^{(n)}):H^{(n)}];\end{matrix}\right.$$

$$\left. u_1\sin^{2\eta_1}\pi x/L,\ldots,u_n\sin^{2\eta_n}\pi x/L\right).$$

Then solution (8.6.10) of the problem II is given by

$$(8.9.12)\quad u(x,t)=\frac{\pi}{2^{W-3}}\sum_{m=1}^{\infty}\frac{\lambda_m \sin(\pi\lambda_m x/L)\cos(\pi\lambda_m ct/L)\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}$$

$$\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i s_i}A_{n_i,s_i}}{4^{\rho_i s_i}}\frac{y_1^{s_1}}{s_1!}\ F^{A+1:B^1;\ldots;B^{(n)}}_{C+2,:D^1;\ldots;D^{(n)}}$$

$$\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(n)}],[W+2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1,\ldots,2\xi_n]: & [1-(b'):\phi'];\ldots;[1-(b^{(n)}):\phi^{(n)}];\\ [1-(c):\psi',\ldots,\psi^{(n)}],\left[\frac{1+W+2(\rho_1 s_1+\ldots+\rho_r s_r)\pm\lambda_m}{2}:\xi_1,\ldots,\xi_n\right]: & [1-(d'):\delta'];\ldots;[1-(d^{(n)}):\delta^{(n)}];\end{matrix} z_1/4^{\xi_1}\ldots z_n/4^{\xi_n}\right)$$

$$+\frac{\pi L}{c.2^{W'-3}}\sum_{m=1}^{\infty}\frac{\sin(\pi\lambda_m x/L)\sin(\pi\lambda_m ct/L)\sin(\pi\lambda_m/2)}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}\sum_{s_1=0}^{[p_1/q_1]}\cdots\sum_{s_l=0}^{[p_l/q_l]}$$

$$\prod_{i=1}^{l} \frac{(-p_i)_{q_i s_i} A_{p_i,s_i}}{4^{\sigma_i s_i}} \frac{x_i^{s_i}}{s_i!} F^{v+1:P^1;\ldots;P^{(n)}}_{W+2,:Q^1;\ldots;Q^{(n)}} \left(\begin{matrix} [1-(e):E',\ldots,E^{(n)}],[1+W'+2(\sigma_1 s_1+\ldots+\sigma_l s_l):2\eta_1,\ldots,2\eta_n]: \\ [1-(g):G',\ldots,G^{(n)}],\left[\frac{1+W'+2(\sigma_1 s_1+\ldots+\sigma_l s_l)\pm\lambda_m}{2}:\eta_1,\ldots,\eta_n\right]: \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(f'):F'];\ldots;[1-(f^{(n)}):F^{(n)}]; \\ [1-(h'):H'];\ldots;[1-(h^{(n)}):H^{(n)}]; \end{matrix} 4^{-\eta_1} u_1 \ldots 4^{-\eta_n} u_n \right)$$

valid if all the conditions of (8.6.10) are satisfied

Similarly specialising the coefficients A_{n_i,s_i} of polynomials $S_{n_i}^{m_i}[x]$ [6] and parameters of H-function of several complex variables ([10],[11], [12]), we shall get a very large number of results involving various polynomials others special functions useful in mathematical analysis, applied mathematics and mathematical physics.

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APPLICATIONS OF GENERALIZED POLYNOMIALS OF SRIVASTAVA AND MULTIVARIABLE *H*-FUNCTION OF SRIVASTAVA-PANDA

CHAPTER -IX

APPLICATIONS OF GENERALIZED POLYNOMALS OF SRIVASTAVA AND MULTIVARIABLE *H*-FUNCTION OF SRIVASTAVA - PANDA

In the present Chapter, we discuss the problem on heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends in side the medium). Its special cases are also discussed.

It is remarked that all the results due to Chaurasia- Gupta ([4],(2.1),(2.2), (3.1), (3.3), (4.1) to (4.12)) are wrongly expressed and their correct forms are special cases of our results. Our results are also generalization of the results due to Chaurasia and Patni [3]. It is also remarked that their all results ([3], (8), (11) to (19)) are also wrongly expressed.

9.1 Introduction. Chandel and Tiwari [1] discussed two boundary value problems with applications of multiple hypergeometric function of Srivastava and Daoust ([6], [7],[8]). Chaurasia and Patni [3] discussed a heat conduction problem and certain product of the multivariable H-function due to Srivastava and Panda ([9], [10], also see [11]) and two general classes of polynomials due to Srivastava [5].Chaurasia and Gupta [4] also discussed a solution of the partial differential equation of

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heat conduction in rod under the Robin condition. Srivastava and Srivastava [15] dicussed application of Meiger's G-funcrtion of one variale in two boundary value problems. In previous Chapter VIII (also see Chandel and Sengar [2]) we discussed two boundary value problems employing multivariable H-function of Srivastava and Panda ([9],[10] also see [11]) and product of several general classes of poynomials of Srivastava [15].

In the present Chapter, we shall discuss the problem on heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends the medium). If the thermal coefficient is constant and there is no source of thermal enegry, then we shall find the function $\theta(x,t)$ satiafying partial differatial equation

$$(9.1.1)\ \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} = t \geq 0$$

with intitial condition

$$(9.1.2)\ u(x,0) = f(x),$$

and boundary conditions

$$(9.1.3)\ \frac{\partial u(0,t)}{\partial t} - hu(0,t) = 0\,,$$

$$(9.1.4)\ \frac{\partial}{\partial t} u(L,t) + hu(L,t) = 0, h > 0.$$

It is clear that the expression

$$(9.1.5)\ u(x,t) = e^{-uE_p^2 t}(m \cos E_p x + n \sin E_p x)$$

satisfies (9.1.1).

Then boundary conditions (9.1.3) and (9.1.4) give respectively

(9.1.6) $m/n = E_p/h$

and

(9.1.7) $tan(E_pL) = 2E_ph/(E_p^2 - h^2)$

where E_p is p^{th} positive root of (9.1.7).

Then the elegant form of solution of the problem is given by

$$(9.1.8)\ u(x,t) = \sum_{p=1}^{\infty} R_p(\cos E_px + h/E_p \sin E_px)e^{-\mu E_p^2 t}.$$

Here we consider the problem of determining $u(x,t)$, where

$$(9.1.9)\ u(x,0) = f(x) = \sin(\pi x/L))^{W-1} S_{n_1}^{m_1}[y_1(\sin \pi x/L)^{2\rho_1}]$$

$$\ldots S_{n_r}^{m_r}[y_r(Sin\,\pi x/L)^{2\rho_r}].H,$$

$$(9.1.10)\ H = H^{0,\lambda:(\mu',\nu');\ldots\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:(B',D');\ldots\ldots;(B^{(n)},D^{(n)})}\left(\begin{matrix}[(a):\theta';\ldots\ldots;\theta^{(n)}]: \\ [(c):\psi';\ldots\ldots;\psi^{(n)}]:\end{matrix}\right.\left(\begin{matrix}[(a):\theta';\ldots\ldots;\theta^{(n)}]: \\ [(c):\psi';\ldots\ldots;\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'),\phi'];\ldots\ldots;[(b^{(n)}),\phi^{(n)}]; \\ [(d'),\delta'];\ldots\ldots;[(d^{(n)}),\delta^{(n)}];\end{matrix} z_1(sin\,\pi x/L)^{2\xi_1};\ldots\ldots;z_n(sin\,\pi x/L)^{2\xi_n}\right)$$

is multivariable H-function of Srivastava and Panda ([9], [10] and also see [11] and $S_n^m[x]$ are generalized polynomids of Srivastava ([5], p.l, eqn. (1.1) difined by

$$(9.1.11)\ S_n^m[x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} A_{n,s}x^s, n = 0,1,2\ldots$$

where m is arbitrary positive integer and the coefficients $A_{n,s}(n,s \geq 0)$ are arbitrary constants, real or complex.

9.2. Formulae Required. The following well known results will be frequently used in our investigations :

$$(9.2.1)\quad \int_0^L (\sin \pi x/L)^{\sigma-1}\sin(E_p\pi x/L)dx = L\,2^{1-\sigma}\sin(E_p\pi/2)\frac{\Gamma(\sigma)}{\Gamma((\sigma\pm E_p+1)/2)}$$

$$(9.2.2)\quad \int_0^L (\sin \pi x/L)^{\sigma-1}\cos(E_p\pi x/L)dx$$

$$= L\,2^{1-\sigma}\cos(E_p\pi/2)\frac{\Gamma(\sigma)}{\Gamma((\sigma\pm E_p+1)/2)}, \qquad Re(\sigma)>0,$$

$$(9.2.3)\quad \int_0^L (\cos E_q x + h/E_q \sin E_q x)(\cos E_p x + h/E_p \sin E_p x)dx$$

$$= [2E_q^2]^{-1}(E_q^2+h^2)L+2h],\ p=q$$

$$=0,\ p\neq q,$$

where E_q is qth positive root of non-algebraic equation

$$(9.2.4)\quad tan\,EL = 2hE/(E^2-h^2).$$

9.3. Main integrals. In this section, we evaluate the following integrals, which will be used in our investigations:

$$(9.3.1)\quad \int_0^L (\sin \pi x/L)^{W-1}\cos(\pi x\lambda_m/L)\prod_{i=1}^{r} S_{n_i}^{m_i}[y_i(\sin \pi x/L)^{2\rho_i}]H dx$$

$$= \frac{L\cos(\pi\lambda_m/2)}{2^{W-1}}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_i s_i}A_{n_i,s_i}y_i^{s_i}}{4\rho_i^{s_i}s_i!}H(s_1,\ldots,s_r;\lambda_m)$$

where

$$H(s_1,\ldots,s_n;\lambda_m) = H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\upsilon^{(n)})[(a):\theta';\ldots;\theta^{(n)}],}_{A+1,C+2:(B',D');\ldots;(B^{(n)},D^{(n)})[(c):\psi';\ldots;\psi^{(n)}],}$$

$$\left(\begin{array}{l}[1-W-2(\rho_1 s_1+\ldots+\rho_r s_r):2\xi_1\ldots,2\xi_n]:[(b'):(\phi')];\ldots;[(b^{(n)}):(\phi^{(n)})],\\ [\{1-W-2(\rho_1 s_1+\ldots\rho_r s_r)\pm\lambda_m\}/2:\xi_1,\ldots,\xi_n]:[(d'):(\delta')];\ldots;[(d^{(n)}):(\delta^{(n)})],\end{array}\right.$$

$$\left. z_1/4^{\xi_1},\ldots,z_n/4^{\xi_n}\right),$$

$\mathrm{Re}(W)>0, \mathrm{Re}(W+\sum_{i=1}^{n}\xi_i d_j^{(i)}/\delta_j^{(i)})>0$, all ξ_i are real positive numbers λ, A, $C, \mu^{(i)}, \upsilon^{(i)}, B^{(i)}, D^{(i)}$ are such that $A\geq\lambda\geq 0, C>0, D^{(i)}\geq\mu^{(i)}\geq 0$, $B^{(i)}\geq\upsilon^{(i)}\geq 0$ and $\theta_j^{(i)}, j=i,\ldots A; \phi_j^{(i)}, j=1,\ldots C; \delta_j^{(i)}, j=1,\ldots B^{(i)}; \psi_j^{(i)}, j=1,\ldots D^{(i)}$ are positive real numbers,

$|\arg[z_i(\sin\pi x/L)^{2\xi_i}]|<\Delta_i\pi/2,$

$$\Delta_i=-\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0; i=1,\ldots,n.$$

Also ρ_k are all positive numbers, n_k, m_k, are arbitrary positive integers, A_{n_k,s_k} are arbitrary functions of n_k and s_k real or complex independent of x, y_k; $\rho_k k=1,\ldots,r$.

For $r=2$, (9.3.1) reduces to the correct form of the result due to Chaurasia and Gupta ([4],(2.2)).

We shall also frequently use the integral due to authors ([2],(2.1))

9.4. Solution of the Problem.

For $t=0$, from (9.1.8) and (9.1.9) we derive

(9.4.1) $$(\sin\pi x/L)^{W-1}\prod_{i=1}^{r}S_{n_i}^{m_i}[y_i(\sin\pi x/L)^{2\rho_i}]H$$

$$= \sum_{q=1}^{\infty} R_q (\cos E_q x + h / E_q \sin E_q x).$$

Then

$$(9.4.2) \int_0^L \sum_{q=1}^{r} R_q (\cos E_q x + h / E_q x \sin E_q x)(\cos E_p + h / E_p \sin E_p x) dx$$

$$= \int_0^L (\sin \pi x / L)^{W-1} (\cos E_p x + h / E_p \sin E_p x \prod_{i=1}^{r} S_{n_i}^{m_i} [y_i (\sin \pi x / L)^{2\rho_i}].$$

Thus making an appeal to (9.2.1), (9.2.2) and (9.2.3), we derive

$$(9.4.3) R_p = \frac{L[E_p \cos(LE_p / 2)] + h \sin(LE_p / 2)] E_p}{2^{W-2} \left[(E_p^2 + h^2) L + 2h \right]}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i, s_i} y_i^{s_i}}{4^{\rho_i s_i} s_i!} H(s_1, \ldots, s_r; LE_p / \pi).$$

Therefore, substituting the value of R_p in (9.1.8), the solution of the problem is given by

$$(9.4.4) \quad u(x,t) = \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p \cos(LE_p / 2)] + h \sin(LE_p / 2)]}{\left[(E_p^2 + h^2) L + 2h \right]}$$

$$E_p (\cos E_p x + h \sin E_p x) e^{-\mu E_p^2 t} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i s_i} y_i^{s_i}}{4^{\rho_i s_i} s_i!}$$

$$H(s_1, \ldots, s_r; \ LE_p/\pi),$$

where all the conditions of (9.3.1) are satisfied.

9.5 Expansion formula. For $t = 0$, (9.4.4) with (9.1.9), gives expansion formula

$$(9.5.1) (\sin \pi x / L)^{W-1} \prod_{i=1}^{r} S_{n_i}^{m_i} [y_i (\sin \pi x / L)]^{2\rho_i} H$$

$$= \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p \cos(LE_p/2) + h\sin(LE_p/2)][E_p \cos E_p x + h \sin E_p x]}{[(E_p^2 + h^2)L + 2h]}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} A_{n_i,s_i} y_i^{s_i}}{4^{\rho_i s_i} s!} H(s_1,\ldots,s_r; LE_p/\pi),$$

valid if all the conditions of (9.3.1) are satisfied.

9.6. Special Cases.

Case I. Choosing each $m_i = 2, A_{n_i s_i} = (-1)^{s_i}$, we have

$$S_{n_i}^{2}[y_i] \to y_i^{n_i/2} H_{n_i}(1/2\sqrt{y_i}), i = 1,\ldots,r.$$

Consider $y_i \equiv y_i (\sin \pi x / L)^{2\rho_i}$

That is $S_{n_i}^{2}[y_i] \to y_i^{n_i/2} (\sin \frac{\pi x}{L})^{\rho_i n_i} H_{n_i}\left(\frac{1}{2\sqrt{y_i}(\sin \pi x/L)^{\rho_i}} \right).$

Therefore, for Hermite polynomials ([14], p.106,eq.(5.5.4) and [12], p. 158), our main integral (9.3.1) reduces to

$$(9.6.1) \int_0^L (\sin\pi x/L)^{W-1} (\cos\pi x\lambda_m/L) \prod_{i=1}^{r} [y_i^{n_i/2} (\sin\pi x\ L)^{\rho_i n_i} . H_{n_i}\left(\frac{1}{2\sqrt{y_i}(\sin\pi x/L)^{\rho_i}} \right) H dx$$

$$= \frac{L\cos(\pi\lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \prod_{i=1}^{r} \frac{(-n_i)_{2s_i} (-1)^{s_i} y_i^{s_i}}{4^{\rho_i s_i} s_i!} H(s_1,\ldots,s_r : \lambda_m)$$

where all conditions of (9.3.1) are satisfied.

Then solution (9.4.3) of the problem reduces to

$$(9.6.2) \quad u(x,t) = \frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p \cos(LE_p/2)] + h\sin(LE_p/2)]}{[(E_p^2 + h^2)L + 2h]}$$

$$[E_p \cos E_p x + h \sin E_p x] e^{-\mu E_p^2 t}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i s_i} (-1)^{s_i} y_i^{s_i}}{4^{\rho_i s_i} s_i!} H(s_1,\ldots,s_r;LE_p/\pi),$$

which is valid if all conditions of (9.3.1) are satisfied.

The expansion formula (9.5.1) reduces to

$$(9.6.3)\quad (\sin \pi x/L)^{W-1} \prod_{i=1}^{r} (\sin \pi x/L)^{n_i \rho_i} y_i^{n_i/2} H_{n_i}\left[\frac{1}{2\sqrt{y_i (\sin \pi x/L)^{\rho_i}}}\right].H$$

$$\frac{L}{2^{W-2}} \sum_{p=1}^{\infty} \frac{[E_p \cos(LE_p/2)] + h \sin(LE_p/2)][E_p \cos E_p x + h \sin E_p x]}{\left[(E_p^2 + h^2)L + 2h\right]}$$

$$\sum_{s_1=0}^{[n_1/2]} \cdots \sum_{s_r=0}^{[n_r/2]} \prod_{i=1}^{r} \frac{(-n_i)_{2s_i} y_i^{s_i} (-1)^{s_i}}{4^{\rho_i s_i} s_i!} H(s_1,\ldots,s_r;LE_p/\pi).$$

Case II. Choosing each $m_i = 1, A_{n_i,s_i} = \binom{n_i+\alpha_i}{n_i} \frac{1}{(1+\alpha_i)_{s_i}}$,

we have $S_{n_i}^{(1)}[y_i] \to L_{n_i}^{(\alpha)}(y_i), i = 1,\ldots,r,$

therefore for Laguerre polynomials ([14], p. 10,eq.[5.1.6] and [12].p.158), our result (9.3.1) reduces to

$$(9.6.4) \int_0^L (\sin \pi x/L)^{W-1} (\cos \pi x \lambda_m/L) \prod_{i=1}^{r} L_{n_i}^{(\alpha_i)}[y_i (\sin \pi x/L)^{2\rho_i}] H dx$$

$$= \frac{L \cos(\pi \lambda_m/2)}{2^{W-1}} \sum_{s_1=0}^{n_1} \cdots \sum_{s_r=0}^{n_r} \prod_{i=1}^{r} \frac{(-n_i)_{s_i} y_i^{s_i}}{4^{\rho_i s_i} s_i!} \binom{n_i+a_i}{n_i} \frac{1}{(1+\alpha_i)_{s_i}} H(s_1,\ldots,s_r;\lambda_m)$$

and the solution (9.4.3) of the problem reduces to

$$(9.6.5)\; u(x,t)=\frac{L}{2^{W-2}}\sum_{p=1}^{\infty}\frac{[E_p\cos(LE_p/2)]+h\sin(LE_p/2)]}{\left[(E_p^2+h^2)L+2h\right]}$$

$$[E_p\cos E_px+h\sin E_px]e^{-\mu E_p^2 t}\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\prod_{i=1}^{r}\binom{n_i+\alpha_i}{n_i}\frac{1}{(1+\alpha_i)_{s_i}}$$

$$\frac{(-n_i)_{s_i}\,y_i^{s_i}(-1)^{s_i}}{4^{\rho_i s_i}\,s_i!}\;H\left(s_1,\ldots,s_r;LE_p/\pi\right),$$

provided that all the conditions of (9.3.1) are satisfied.

Then expansion formula (9.5.1) reduces to

$$(9.6.6)\quad (\sin\pi x/L)^{W-1}\prod_{i=1}^{r}L_{n_i}^{(\alpha_i)}[y_i(\sin\pi x/L)^{2\rho_i}].H$$

$$=\frac{L}{2^{W-2}}\sum_{p=1}^{\infty}\frac{\left[E_p\cos(LE_p/2)+h\sin(LE_p/2)\right]\left[E_pCosE_px+h\sin E_px\right]}{\left[(E_p^2+h^2)L+2h\right]}$$

$$\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\prod_{i=1}^{r}\frac{(-n_i)_{s_i}}{4^{\rho_i s_i}s_i!}\binom{n_i+\alpha_i}{n_i}\frac{y_i^{s_i}}{(1+\alpha_i)_{s_i}}\;H\left(s_1,\ldots,s_r;LE_p/\pi\right).$$

Case III. Choosing each $m_i=1, A_{n_i,s_i}=\binom{n_i+\alpha_i}{n_i}\dfrac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}$

we have $S_{n_i}^{(1)}[y_i]\rightarrow P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i), i=1,\ldots,r.$

Therefore, for Jacobi polynomials ([14],p.68,eq. (4.3.2) and [12] p.159),(9.3.1) reduces to

$$(9.6.7)\int_0^L (\sin\pi x/L)^{W-1}(\cos\pi x\lambda_m/L)\prod_{i=1}^{r}P_{n_i}^{(\alpha_i,\beta_i)}[1-2y_i(\sin\pi x/L)^{2\rho_i}]$$

$$=\frac{L\cos(\pi\lambda_m/2)}{2^{-1}}\sum_{s_i=0}^{n_i}\cdots\sum_{s_r=0}^{n_r}\prod_{i=1}^{r}\frac{(-n_i)_{s_i}y_i^{s_i}}{4^{\rho_i s_i}s_i!}\binom{n_i+\alpha_i}{n_i}$$

$$\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}} H(s_1,\ldots,s_r;\lambda_m)$$

and the solution (9.4.3) of the problem reduces to

(9.6.8) $$u(x,t)=\frac{L}{2^{W-2}}\sum_{p=1}^{\infty}\frac{[E_p\cos(LE_p/2)]+h\sin(LE_p/2)]}{\left[(E_p^2+h^2)L+2h\right]}$$

$$[E_p\cos E_p x+h\sin E_p x]e^{-\mu E_p^2 t}\sum_{s_1=0}^{n_1}\cdots\sum_{s_r=0}^{n_r}\prod_{i=1}^{r}\frac{(-n_i)_{s_i}y_i^{s_i}}{4^{\rho_i s_i}s_i!}$$

$$\binom{n_i+\alpha_i}{n_i}\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}} H(s_1,\ldots,s_r;LE_p/\pi),$$

where all conditions of (9.3.1) are satisfied.

Then expansion formula (9.5.1) reduces to

(9.6.9) $$(\sin\pi x/L)^{W-1}\prod_{i=1}^{r} P_{n_i}^{(\alpha_i,\beta_i)}[1-2y_i(\sin\pi x/L)^{2\rho_i}].H$$

$$=\frac{L}{2^{W-2}}\sum_{p=1}^{\infty}\frac{[E_p\cos(LE_p/2)]+h\sin(LE_p/2)][E_p\cos E_p x+h\sin E_p x]}{\left[(E_p^2+h^2)L+2h\right]}$$

$$\sum_{s_1=0}^{n}\cdots\sum_{s_r=0}^{n}\frac{(-n_i)_{s_i}}{4^{\rho_i s_i}s_i!}\binom{n_i+\alpha_i}{n_i}\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}y_i^{s_i}}{(1+\alpha_i)_{s_i}} H(s_1,\ldots,s_r;LE_p/\pi).$$

valid if all conditions of (9.3.1) are satisfied.

9.7. Remarks.

Remark 1. Specially for $r=2$, our results (9.3.1),(9.4.2), (9.4.3), (9.4.4), (9.6.1), (6.2),(6.4) and (6.5) reduce to correct forms of wrongly expressed results due to Chaurasia and Gupta ([4],(2.2) ,(3.2), (3.3),(3.1), (4.5),(4.6),(4.11) and (4.12)) respectively.

Remark 2. For $r=2, \lambda=A, u^{(i)}=1, \upsilon^{(i)}=B^{(i)}$ and replacing $D^{(i)}$ by $D^{(i)}+1$, $i=1,\ldots,n$; our results (9.3.1) and (9.4.4) reduce to correct forms of the

results due to Chaurasia and Gupta ([4],(4.8) and (4.9)) respectively.

Remark 3. For $r=2, n=1$, our results (9.3.1) and (9.4.4) reduce to correct forms of ([4],(4.2) and (4.3)) respectively.

Remark 4. Here it is also remarked that all the remaining results due to Chaurasia and Gupta ([4],(2.1),(4.1),(4.4),(4.7) and (4.10)) are wrongly expressed.

Remarks 5. The above remarks also suggest that all the results due to Chaurasia and Patni ([3],(8),(11) to (19)) are also worngly expressed.

Actually every where $2^{-hs-h's'}$ should be written within $\sum_{s=0}^{[n/m]} \sum_{s'=0}^{[n'/m']}$.

Specializing the parameters of multivariable H-runction of Srivastava and Panda ([8],[9], also see [10]) and parameters and number of the polynomials [5], we can derive serveral interesting results very useful in Analysis, Applied Mathematics and Mathematical Physics.

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